APPENDIX A

Elementary Matrix Theory and Algebra

A-1 ELEMENTARY MATRIX THEORY

In the study of modern control theory, it is often desirable to use matrix notation to simplify complex mathematical expressions. The matrix notation usually makes the equations much easier to handle and manipulate.

As a motivation to the reason of using matrix notation, let us consider the following set of n simultaneous algebraic equations:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n \end{array} \tag{A-1}$$

We may use the matrix equation

$$\mathbf{Ax} = \mathbf{y} \tag{A-2}$$

as a simplified representation of Eq. (A-1). The symbols **A**, **x**, and **y** are defined as **matrices**, which contain the coefficients and variables of the original equations as their elements. In terms of matrix algebra, which will be discussed shortly, Eq. (A-2) can be stated as *the product of the matrix* **A** *and* vector **x** *is equal to the* vector **y**. The three matrices involved are defined as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{A-3}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{A-4}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{A-5}$$

which are simply bracketed arrays of coefficients and variables. These examples of matrices prompted the following definition of a matrix.

A-1-1 Definition of a Matrix

A matrix is a collection of elements arranged in a rectangular or square array. There are several ways of bracketing a matrix. In this text, the square brackets, such as those in Eqs. (A-3) through (A-5), are used to represent matrices. It is important to distinguish between a **matrix** and a **determinant**. The basic characteristics of these are listed as follows:

Matrix	Determinant
• An array of numbers or elements with n rows and m columns. • Does not have a value, although a square matrix ($n = m$) has a determinant.	• An array of numbers or elements with n rows and n columns (always square). • Has a value.

Some important definitions of matrices are given in the following paragraphs.

Matrix Elements: When a matrix is written

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{A-6}$$

where a_{ij} is defined as the element in the ith **row** and the jth **column** of the matrix. As a rule, we always refer to the row first and the column last.

Order of a Matrix: *The order of a matrix refers to the total number of rows and columns of the matrix*. For example, the matrix in Eq. (A-6) has three rows and three columns and is called a 3×3 (three-by-three) matrix. A matrix with n rows and m columns is termed $n \times m$, or n by m.

Square Matrix: *A square matrix is one that has the same number of rows as columns.*

Column Matrix: *A column matrix is one that has one column and more than one row, that is, an $m \times 1$ matrix, $m > 1$*. Quite often, a column matrix is referred to as a **column vector** or simply an **m-vector** if there are m rows and one column. The matrix in Eq. (A-4) is a typical n-vector.

Row Matrix: *A row matrix is one that has one row and more than one column, that is, a $1 \times n$ matrix, where $n > 1$*. A row matrix can also be referred as a **row-vector**.

Diagonal Matrix: *A diagonal matrix is a square matrix with $a_{ij} = 0$ for all $i \neq j$.* Examples of a diagonal matrix are

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \quad \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

Unity Matrix (Identity Matrix): *A unity matrix is a diagonal matrix with all the elements on the main diagonal ($i = j$) equal to 1*. A unity matrix is often designated by

I or **U**. An example of a unity matrix is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A-7}$$

Null Matrix: *A null matrix is one whose elements are all equal to zero.*

Symmetric Matrix: *A symmetric matrix is a square matrix that satisfies the condition $a_{ij} = a_{ji}$ for all i and j.* A symmetric matrix has the property that, if its rows are interchanged with its columns, the same matrix is obtained. Two examples of the symmetric matrix are

$$\mathbf{A} = \begin{bmatrix} 6 & 5 & 1 \\ 5 & 0 & 10 \\ 1 & 10 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & -4 \\ -4 & 1 \end{bmatrix} \tag{A-8}$$

Determinant of a Matrix: *With each square matrix, a determinant having the same elements and order as the matrix may be defined. The determinant of a square matrix* **A** *is designated by*

$$\det \mathbf{A} = \Delta_{\mathrm{A}} = |\mathbf{A}| \tag{A-9}$$

For example, the determinant of the matrix in Eq. (A-6) is

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{A-10}$$

Cofactor of a Determinant Element: *Given any* n*th-order determinant* $|\mathbf{A}|$, *the cofactor A_{ij} of any element a_{ij} is the determinant obtained by eliminating all elements of the ith row and jth column and then multiplied by* $(-1)^{i+j}$. For example, the cofactor of the element a_{11} of $|\mathbf{A}|$ in Eq. (A-10) is

$$A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32} \tag{A-11}$$

In general, the value of a determinant can be written in terms of the cofactors. Let **A** be an $n \times n$ matrix, then the determinant of **A** can be written in terms of the cofactor of any row or the cofactor of any column. That is,

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{ij}A_{ij} \quad (i = 1, \text{ or } 2, \ldots, \text{or } n) \tag{A-12}$$

or

$$\det \mathbf{A} = \sum_{i=1}^{n} a_{ij}A_{ij} \quad (j = 1, \text{or } 2, \ldots, \text{or } n) \tag{A-13}$$

▶ **EXAMPLE A-1-1** The value of the determinant in Eq. (A-10) is

$$\begin{aligned} \det \mathbf{A} = |\mathbf{A}| &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned} \tag{A-14}$$

Singular Matrix: *A square matrix is said to be singular if the value of its determinant is zero*. If a square matrix has a nonzero determinant, it is called a **nonsingular matrix**. When a matrix is singular, it usually means that not all the rows or not all the columns of the matrix are independent of each other. When the matrix is used to represent a set of algebraic equations, singularity of the matrix means that these equations are not independent of each other. ◀

▶ **EXAMPLE A-1-2** Consider the following set of equations:

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 0 \\ -x_1 + x_2 + x_3 &= 0 \\ x_1 - 2x_2 + 2x_3 &= 0 \end{aligned} \tag{A-15}$$

The third equation is equal to the sum of the first two. Thus, these three equations are not completely independent. In matrix form, these equations may be written as

$$\mathbf{Ax} = 0 \tag{A-16}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 1 & 1 \\ 1 & -2 & 2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{A-17}$$

and $\mathbf{0}$ is a 3×1 null vector. The determinant of $\mathbf{A}$ is 0, and, thus, the matrix $\mathbf{A}$ is singular. In this case, the rows of $\mathbf{A}$ are dependent.

Transpose of a Matrix: *The transpose of a matrix* $\mathbf{A}$ *is defined as the matrix that is obtained by interchanging the corresponding rows and columns in* $\mathbf{A}$. Let $\mathbf{A}$ be an $n \times m$ matrix that is represented by

$$\mathbf{A} = [a_{ij}]_{n,m} \tag{A-18}$$

The transpose of $\mathbf{A}$, denoted by $\mathbf{A}'$, is given by

$$\mathbf{A}' = \text{transpose of } \mathbf{A} = [a_{ji}]_{m,n} \tag{A-19}$$

Notice that the order of $\mathbf{A}$ is $n \times m$, but the order of $\mathbf{A}'$ is $m \times n$. ◀

▶ **EXAMPLE A-1-3** Consider the 2×3 matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 5 \end{bmatrix} \tag{A-20}$$

The transpose of $\mathbf{A}$ is obtained by interchanging the rows and the columns.

$$\mathbf{A}' = \begin{bmatrix} 3 & 0 \\ 2 & -1 \\ 1 & 5 \end{bmatrix} \tag{A-21}$$

◀

Some Properties of Matrix Transpose

1. $(\mathbf{A}')' = \mathbf{A}$ (A-22)
2. $(k\mathbf{A})' = k\mathbf{A}'$, where k is a scalar (A-23)
3. $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$ (A-24)
4. $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$ (A-25)

Adjoint of a Matrix: *Let* **A** *be a square matrix of order n. The adjoint matrix of* **A**, *denoted by* adj **A**, *is defined as*

$$\text{adj}\,\mathbf{A} = \left[A_{ij} \text{ of } \det \mathbf{A}\right]'_{n,n} \tag{A-26}$$

where A_{ij} denotes the cofactor of a_{ij}.

▶ **EXAMPLE A-1-4** Consider the 2 × 2 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{A-27}$$

The cofactors are $A_{11} = a_{22}$, $A_{12} = -a_{21}$, $A_{21} = -a_{12}$, and $A_{22} = a_{11}$. Thus, the adjoint matrix of **A** is

$$\text{adj}\,\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}' = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}' = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{A-28}$$

Trace of a Square Matrix: *Given an n × n matrix with elements a_{ij}, the* **trace of A**, *denoted as* tr (**A**), *is defined as the sum of the elements on the main diagonal of* **A**; *that is*

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} \tag{A-29}$$

The trace of a matrix has the following properties:

1. $\text{tr}(\mathbf{A}') = \text{tr}(\mathbf{A})$ (A-30)
2. For $n \times n$ square matrices **A** and **B**,

$$\text{tr}(\mathbf{A}+\mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B}) \tag{A-31}$$

◀

▶ A-2 MATRIX ALGEBRA

When carrying out matrix operations, it is necessary to define matrix algebra in the form of addition, subtraction, multiplication, and division.

A-2-1 Equality of Matrices

Two matrices **A** and **B** are said to be equal to each other if they satisfy the following conditions:

1. They are of the same order.
2. The corresponding elements are equal; that is,

$$a_{ij} = b_{ij} \quad \text{for every } i \text{ and } j \tag{A-32}$$

▶ **EXAMPLE A-2-1**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{A-33}$$

implies that $a_{11} = b_{11}$, $a_{12} = b_{12}$, $a_{21} = b_{21}$, $a_{22} = b_{22}$. ◀

A-2-2 Addition and Subtraction of Matrices

Two matrices **A** and **B** can be added or subtracted to form **A** ± **B** if they are of the same order. That is,

$$\mathbf{A} \pm \mathbf{B} = [a_{ij}]_{n,m} \pm [b_{ij}]_{n,m} = \mathbf{C} = [c_{ij}]_{n,m} \tag{A-34}$$

where

$$c_{ij} = a_{ij} \pm b_{ij} \quad \text{for all } i \text{ and } j. \tag{A-35}$$

The order of the matrices is preserved after addition or subtraction.

▶ **EXAMPLE A-2-2** Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 0 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 3 \\ -1 & 2 \\ 1 & 0 \end{bmatrix} \tag{A-36}$$

which are of the same order. Then the sum of **A** and **B** is

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} 3+0 & 2+3 \\ -1-1 & 4+2 \\ 0+1 & -1+0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & 6 \\ 1 & -1 \end{bmatrix} \tag{A-37}$$

◀

A-2-3 Associative Law of Matrix (Addition and Subtraction)

The associative law of scalar algebra still holds for matrix addition and subtraction. That is,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{A-38}$$

A-2-4 Commutative Law of Matrix (Addition and Subtraction)

The commutative law for matrix addition and subtraction states that the following matrix relationship is true:

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{B} + \mathbf{C} + \mathbf{A} = \mathbf{A} + \mathbf{C} + \mathbf{B} \tag{A-39}$$

as well as other possible commutative combinations.

A-2-5 Matrix Multiplication

The matrices **A** and **B** may be multiplied together to form the product **AB** if they are **conformable**. This means that the number of columns of **A** must equal the number of rows of **B**. In other words, let

$$\mathbf{A} = [a_{ij}]_{n,p} \quad \mathbf{B} = [b_{ij}]_{q,m} \tag{A-40}$$

Then **A** and **B** are conformable to form the product

$$\mathbf{C} = \mathbf{AB} = [a_{ij}]_{n,p}[b_{ij}]_{q,m} = [c_{ij}]_{n,m} \tag{A-41}$$

if and only if $p = q$. The matrix **C** will have the same number of rows as **A** and the same number of columns as **B**.

It is important to note that **A** and **B** may be conformable to form **AB**, but they may not be conformable for the product **BA**, unless in Eq. (A-41), n also equals m. This points out an important fact that *the commutative law is not generally valid for matrix multiplication.* It is also noteworthy that, even though **A** and **B** are conformable for both **AB** and **BA**, usually $\mathbf{AB} \neq \mathbf{BA}$. The following references are made with respect to matrix manipulation whenever they exist:

$$\mathbf{AB} = \mathbf{A}\text{ postmultiplied by }\mathbf{B} \quad \text{or} \quad \mathbf{B}\text{ premultiplied by }\mathbf{A} \tag{A-42}$$

A-2-6 Rules of Matrix Multiplication

When the matrices **A** ($n \times p$) and **B** ($p \times m$) are conformable to form the matrix $\mathbf{C} = \mathbf{AB}$, the ijth element of **C**, c_{ij}, is given by

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \tag{A-43}$$

for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$.

▶ **EXAMPLE A-2-3** Given the matrices

$$\mathbf{A} = [a_{ij}]_{2,3} \quad \mathbf{B} = [b_{ij}]_{3,1} \tag{A-44}$$

the two matrices are conformable for the product **AB** but not for **BA**. Thus,

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \end{bmatrix} \tag{A-45}$$

◀

▶ **EXAMPLE A-2-4** Given the matrices

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \tag{A-46}$$

the two matrices are conformable for **AB** and **BA**.

$$\mathbf{AB} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ 2 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix} \tag{A-47}$$

$$\mathbf{BA} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 6 & -1 \end{bmatrix} \tag{A-48}$$

Therefore, even though **AB** and **BA** both exist, they are not equal. In fact, in this case the products are not of the same order.

Although the commutative law does not hold in general for matrix multiplication, the **associative** and **distributive** laws are valid. For the distributive law, we state that

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \tag{A-49}$$

if the products are conformable. For the associative law,

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \tag{A-50}$$

if the product is conformable. ◀

A-2-7 Multiplication by a Scalar *k*

Multiplying a matrix **A** *by any scalar k is equivalent to multiplying each element of* **A** *by k.*

A-2-8 Inverse of a Matrix (Matrix Division)

In the algebra of scalar quantities, when we write $y = ax$, it implies that $x = y/a$ is also true. In matrix algebra, if $\mathbf{Ax} = \mathbf{y}$, then it *may be possible* to write

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{A-51}$$

where $\mathbf{A}^{-1}$ denotes the **matrix inverse** of **A**. The conditions that $\mathbf{A}^{-1}$ exists are as follows:

1. **A** is a square matrix.
2. **A** must be nonsingular.
3. If $\mathbf{A}^{-1}$ exists, it is given by

$$\mathbf{A}^{-1} = \frac{\text{adj}\,\mathbf{A}}{|\mathbf{A}|} \tag{A-52}$$

▶ **EXAMPLE A-2-5** Given the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{A-53}$$

the inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{\text{adj}\mathbf{A}}{|\mathbf{A}|} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{a_{11}a_{22} - a_{12}a_{21}} \tag{A-54}$$

where for **A** to be nonsingular, $|\mathbf{A}| \neq 0$, or $a_{11}a_{22} - a_{12}a_{21} \neq 0$.
Eq. (A-54) shows that adj **A** of a 2×2 matrix is obtained by *interchanging the two elements on the main diagonal and changing the signs of the elements off the diagonal of* **A**. ◀

▶ **EXAMPLE A-2-6** Given the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{A-55}$$

to find the inverse of **A**, the adjoint of **A** is

$$\text{adj}\,\mathbf{A} = \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & (-a_{12}a_{33} - a_{13}a_{32}) & a_{12}a_{23} - a_{13}a_{22} \\ -(a_{21}a_{33} - a_{23}a_{31}) & a_{11}a_{33} - a_{13}a_{31} & -(a_{11}a_{23} - a_{21}a_{13}) \\ a_{21}a_{32} - a_{22}a_{31} & (-a_{11}a_{32} - a_{12}a_{31}) & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} \tag{A-56}$$

The determinant of **A** is

$$|\mathbf{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \tag{A-57}$$

◀

Some Properties of Matrix Inverse

1. $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ (A-58)
2. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ (A-59)

3. In matrix algebra, in general, $\mathbf{AB} = \mathbf{AC}$ does not necessarily imply that $\mathbf{B} = \mathbf{C}$. The reader can easily construct an example to illustrate this property. However, if $\mathbf{A}$ is a square, nonsingular matrix, we can premultiply both sides of $\mathbf{AB} = \mathbf{AC}$ by $\mathbf{A}^{-1}$. Then,

$$\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{AC} \tag{A-60}$$

which leads to $\mathbf{B} = \mathbf{C}$.

4. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices and are nonsingular, then

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{A-61}$$

A-2-9 Rank of a Matrix

The rank of a matrix $\mathbf{A}$ *is the maximum number of linearly independent columns of* $\mathbf{A}$; *that is, it is the order of the largest nonsingular matrix contained in* $\mathbf{A}$.

EXAMPLE A-2-7 Several examples on the rank of a matrix are as follows:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{rank} = 1 \qquad \begin{bmatrix} 0 & 5 & 1 & 4 \\ 3 & 0 & 3 & 2 \end{bmatrix} \quad \text{rank} = 2$$

$$\begin{bmatrix} 3 & 9 & 2 \\ 1 & 3 & 0 \\ 2 & 6 & 1 \end{bmatrix} \quad \text{rank} = 2 \qquad \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{rank} = 3$$

The following properties are useful in the determination of the rank of a matrix. Given an $n \times m$ matrix $\mathbf{A}$,

1. Rank of $\mathbf{A}'$ = Rank of $\mathbf{A}$.
2. Rank of $\mathbf{A}'\mathbf{A}$ = Rank of $\mathbf{A}$.
3. Rank of $\mathbf{AA}'$ = Rank of $\mathbf{A}$.

Properties 2 and 3 are useful in the determination of rank; because, $\mathbf{A}'\mathbf{A}$ and $\mathbf{AA}'$ are always square, the rank condition can be checked by evaluating the determinant of these matrices.

A-3 COMPUTER-AIDED SOLUTIONS OF MATRICES

Many commercial software packages such as MATLAB, **Maple,** and **MATHCAD** contain routines for matrix manipulations.

REFERENCES

1. R. Bellman, *Introduction to Matrix Analysis,* McGraw-Hill, New York 1960.
2. F. Ayres, Jr., *Theory and Problems of Matrices*, Shaum's Outline Series, McGraw-Hill, New York 1962.