APPENDIX **E**

Properties and Construction of the Root Loci

The following properties of the root loci are useful for constructing the root loci manually and for understanding the root loci. The properties are developed based on the relationship between the poles and zeros of $G(s)H(s)$ and the zeros of $1 + G(s)H(s)$, which are the roots of the characteristic equation.

E-1 $K = 0$ AND $K = \pm\infty$ POINTS

The $K = 0$ points on the root loci are at the poles of $G(s)H(s)$.
The $K = \pm\infty$ points on the root loci are at the zeros of $G(s)H(s)$.
The poles and zeros referred to here include those at infinity, if any.

The reason for this is seen from the condition of the root loci given by Eq. (7-12)

$$G_1(s)H_1(s) = -\frac{1}{K} \tag{E-1}$$

As the magnitude of K approaches zero, $G_1(s)H_1(s)$ approaches infinity, so s must approach the poles of $G_1(s)H_1(s)$ or $G(s)H(s)$. Similarly, as the magnitude of K approaches infinity, s must approach the poles of $G(s)H(s)$.

EXAMPLE E-1-1 Consider the equation

$$s(s+2)(s+3) + K(s+1) = 0 \tag{E-2}$$

When $K = 0$, the three roots of the equation are at $s = 0, -2$, and -3. When the magnitude of K is infinite, the three roots of the equation are at $s = -1, \infty$ and ∞. It is useful to consider that infinity in the s-plane is a point concept. We can visualize that the finite s-plane is only a small portion of a sphere with an infinite radius. Then, infinity in the s-plane is a point on the opposite side of the sphere that we face.

Dividing both sides of Eq. (E-2) by the terms that do not contain K, we get

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s+2)(s+3)} = 0 \tag{E-3}$$

which gives

$$G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)} \tag{E-4}$$

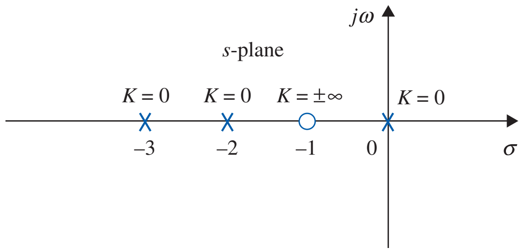

Figure E-1 Roots at which $K = 0$ on the root loci of $s(s+2)(s+3) + K(s+1) = 0$.

Thus, the three roots of Eq. (E-2) when $K = 0$ are the same as the poles of the function $G(s)H(s)$. The three roots of Eq. (E-2) when $K = \pm\infty$ are at the three zeros of $G(s)H(s)$, including those at infinity. The three points on the root loci at which $K = 0$ and those at which $K = \pm\infty$ are shown in Fig. E-1. ◀

▶ E-2 NUMBER OF BRANCHES ON THE ROOT LOCI

A branch of the root loci is the locus of one root when K varies between $-\infty$ and ∞. The following property of the root loci results, since the number of branches of the root loci must equal the number of roots of the equation.

The number of branches of the root loci of $F(s) = P(s) + KQ(s) = 0$ is equal to the order of the polynomial.

• It is important to keep track of the total number of branches of the root loci.

Keeping track of the individual branches and the total number of branches of the root locus diagram is important in making certain that the plot is done correctly. This is particularly true when the root locus plot is done by a computer, since unless each root locus branch is coded by a different color, it is up to the user to make the distinctions.

▶ EXAMPLE E-2-1 The number of branches of the root loci of

$$s(s+2)(s+3) + K(s+1) = 0 \tag{E-5}$$

is three, since the equation is of the third order. In other words, the equation has three roots, and thus there should be three root loci. ◀

▶ E-3 SYMMETRY OF THE ROOT LOCI

• It is important to pay attention to the symmetry of the root loci.

The root loci are symmetrical with respect to the real axis of the s-plane. In general, the root loci are symmetrical with respect to the axes of symmetry of the pole-zero configuration of $G(s)H(s)$.

The reason behind this property is because for a polynomial with real coefficients the roots must be real or in complex-conjugate pairs. In general, if the poles and zeros of $G(s)H(s)$ are symmetrical to an axis in addition to the real axis in the s-plane, we can regard this axis of symmetry as if it were the real axis of a new complex plane obtained through a linear transformation.

▶ EXAMPLE E-3-1 Consider the equation

$$s(s+1)(s+2)+K=0 \tag{E-6}$$

Dividing both sides of the equation by the terms that do not contain K, we get

$$G(s)H(s)=\frac{K}{s(s+1)(s+2)} \tag{E-7}$$

The root loci of Eq. (E-6) are shown in Fig. E-2 for $K=-\infty$ to $K=\infty$. Since the pole-zero configuration of $G(s)H(s)$ is symmetrical with respect to the real axis as well as the $s=-1$ axis, the root locus plot is symmetrical to the two axes.

As a review of all the properties of the root loci presented thus far, we conduct the following exercise with regard to the root loci in Fig. E-2.

The points at which $K=0$ are at the poles of $G(s)H(s)$, $s=0, -1$, and -2. The function $G(s)H(s)$ has three zeros at $s=\infty$ at which $K=\pm\infty$. The reader should try to trace out the three separate branches of the root loci by starting from one of the $K=-\infty$ points, through the $K=0$ point on the same branch, and ending at $K=\infty$ at $s=\infty$.

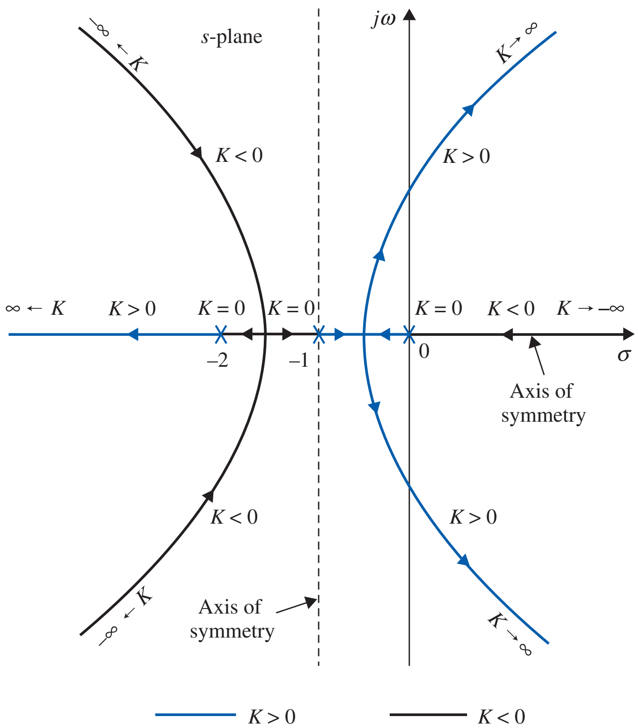

Figure E-2 Root loci of $s(s+2)(s+3)+K(s+1)=0$, showing the properties of symmetry.

▶ **EXAMPLE E-3-2** When the pole-zero configuration of $G(s)H(s)$ is symmetrical with respect to a point in the s-plane, the root loci will also be symmetrical to that point. This is illustrated by the root locus plot of

$$s(s+1)(s+1+j)(s+1-j)+K=0 \tag{E-8}$$

shown in Fig. E-3.

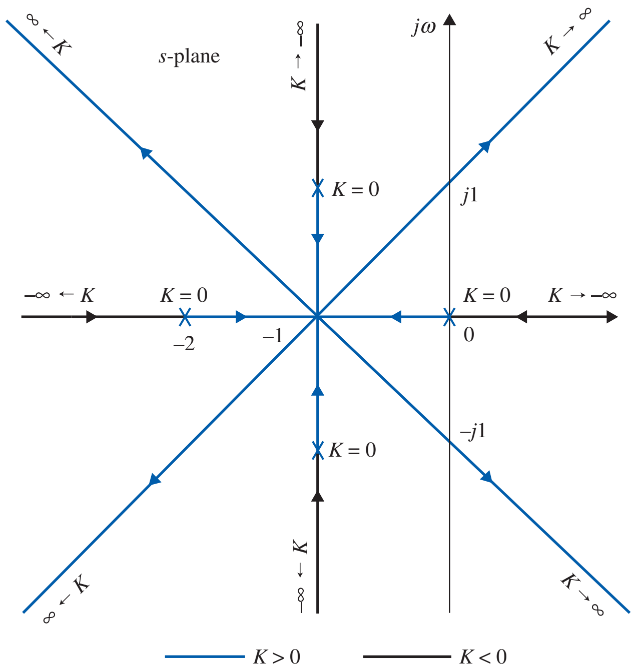

Figure E-3 Root loci of $s(s+2)(s^2+2s+2)+K=0$, showing the properties of symmetry.

◀

▶ E-4 ANGLES OF ASYMPTOTES OF THE ROOT LOCI AND BEHAVIOR OF THE ROOT LOCI AT $|s| = \infty$

• Asymptotes of root loci refers to behavior of root loci at $|s| \rightarrow \infty$.

As shown by the root loci in Figs. E-2 and E-3, when n, the order of $P(s)$ is not equal to m, the order of $Q(s)$, $2|n-m|$ of the loci will approach infinity in the s-plane. The properties of the root loci near infinity in the s-plane are described by the **asymptotes** of the loci when $|s| \rightarrow \infty$. The angles of the asymptotes and their intersect with the real axis of the s-plane are described as follows.

For large values of s, the root loci for $K \geq 0$ (RL) are asymptotic to asymptotes with angles given by

$$\theta_i = \frac{(2i+1)}{|n-m|} \times 180^\circ \qquad n \neq m \tag{E-9}$$

where $i = 0, 1, 2, \ldots, |n-m|-1$; and n and m are the number of finite poles and zeros of $G(s)H(s)$, respectively.

For $K \leq 0$ **(RL),** ***the angles of the asymptotes are***

$$\theta_i = \frac{2i}{|n-m|} \times 180^\circ \qquad n \neq m \tag{E-10}$$

where $\mathbf{i = 0, 1, 2, \ldots, |n - m| - 1}$.

▶ E-5 INTERSECT OF THE ASYMPTOTES (CENTROID)

The intersection of the $2|n-m|$ ***asymptotes of the root loci lies on the real axis of the s-plane, at***

$$\sigma_1 = \frac{\Sigma \text{ finite poles of } G(s)H(s) - \Sigma \text{ finite zeros of } G(s)H(s)}{n-m} \tag{E-11}$$

where n is the number of finite poles and m is the number of finite zeros of G(s)H(s), respectively. The intersection of the asymptotes σ_1 ***represents the center of gravity of the root loci, and is always a real number.***

Since the poles and zeros of $G(s)H(s)$ are either real or in complex-conjugate pairs, the imaginary parts in the numerator of Eq. (7-35) always cancel each other out. Thus, in Eq. (E-11), the terms in the summations may be replaced by the real parts of the poles and zeros of $G(s)H(s)$, respectively. That is,

$$\sigma_1 = \frac{\Sigma \text{ real parts of poles of } G(s)H(s) - \Sigma \text{ real parts of zeros of } G(s)H(s)}{n-m} \tag{E-12}$$

▶ **EXAMPLE E-5-1** Consider the transfer function

$$G(s)H(s) = \frac{K(s+1)}{s(s+4)(s^2+2s+2)} \tag{E-13}$$

which corresponds to the characteristic equation

$$s(s+4)(s^2+2s+2) + K(s+1) = 0 \tag{E-14}$$

The pole-zero configuration of $G(s)H(s)$ is shown in Fig. E-4. From the six properties of the root loci discussed so far, the following information concerning the root loci of Eq. (E-14) when K varies from $-\infty$ to ∞ is obtained:

1. $K = 0$: The points at which $K = 0$ on the root loci are at the poles of $G(s)H(s)$: $s = 0, -4, -1 + j$, and $-1 - j$.
2. $K = \pm\infty$: The points at which $K = \pm\infty$ on the root loci are at the zeros of $G(s)H(s)$: $s = -1, \infty, \infty$, and ∞.
3. There are four root loci branches, since Eqs. (E-13) and (E-14) are of the fourth order.
4. The root loci are symmetrical to the real axis.

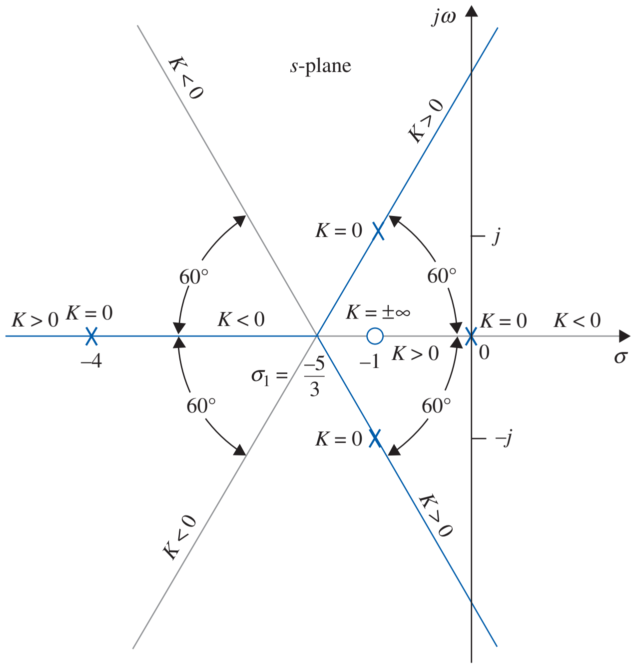

Figure E-4 Asymptotes of the root loci of $s(s+4)(s^2+2s+2)+K(s+1)=0$.

5. Since the number of finite poles of $G(s)H(s)$ exceeds the number of finite zeros of $G(s)H(s)$ by three ($n-m=4-1=3$), when $K=\pm\infty$, three root loci approach $s=\infty$. The angles of the asymptotes of the RL ($K \geq 0$) are given by Eq. (E-9):

$$j=0: \quad \theta_0 = \frac{180^\circ}{3} = 60^\circ$$

$$j=1: \quad \theta_1 = \frac{540^\circ}{3} = 180^\circ$$

$$j=2: \quad \theta_2 = \frac{900^\circ}{3} = 300^\circ$$

The angles of the asymptotes of the root loci for $K \leq 0$ are given by Eq. (E-10), and are calculated to be 0°, 120°, and 240°.

6. The intersection of the asymptotes is given by Eq. (E-12):

$$\sigma_1 = \frac{(-4-1-1)-(-1)}{4-1} = -\frac{5}{3} \tag{E-15}$$

The asymptotes of the root loci are shown in Fig. E-4. ◀

▶ **EXAMPLE E-5-2** The asymptotes of the root loci of several equations are shown in Fig. E-5.

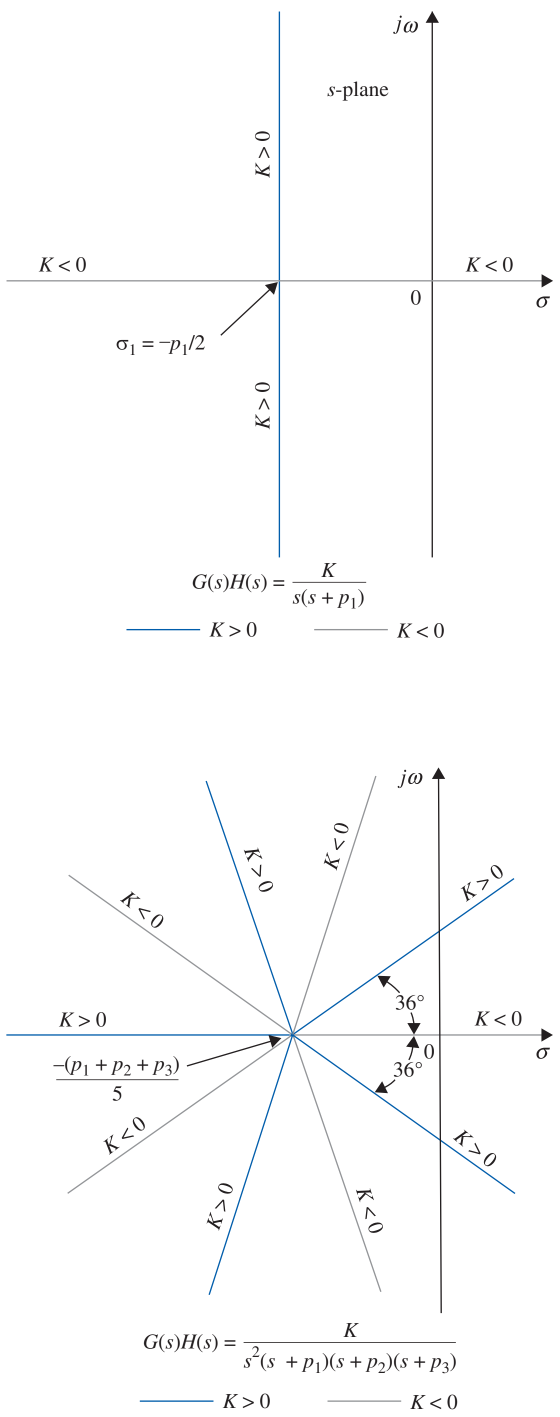

Figure E-5 Examples of the asymtotes of the root loci.

▶ E-6 ROOT LOCI ON THE REAL AXIS

• The entire real axis of the s-plane is occupied by root loci.

The entire real axis of the s-plane is occupied by the root loci for $-\infty \leq K \leq \infty$.

1. ***$K \geq 0$: On a given section of the real axis, root loci are found in the section only if the total number of poles and zeros of G(s)H(s) to the right of the section is odd.***
2. ***$K \leq 0$: On a given section of the real axis, root loci are found in the section only if the total number of real poles and zeros of G(s)H(s) to the right of the section is even. Complex poles and zeros of G(s)H(s) do not affect the type of root loci found on the real axis.***

These properties are arrived at based on the following observations:

1. At any point s_1 on the real axis, the angles of the vectors drawn from the complex-conjugate poles and zeros of $G(s)H(s)$ add up to zero. Thus, only the real zeros and poles of $G(s)H(s)$ contribute to the angular relations in Eqs. (7-18) and (7-19).
2. Only the real poles and zeros of $G(s)H(s)$ that lie to the right of the point s_1 contribute to Eqs. (7-18) and (7-19), because real poles and zeros that lie to the left of the point contribute nothing.
3. Each real pole of $G(s)H(s)$ to the right of s_1 contributes -180 degrees, and each real zero of $G(s)H(s)$ to the right of s_1 contributes $+180$ degrees to Eqs. (7-18) and (7-19).

The last observation shows that for s_1 to be a point on the root locus, there must be an **odd** number of poles and zeros of $G(s)H(s)$ to the right of the point. For s_1 to be a point on the branch of the root loci for $K \leq 0$, the total number of poles and zeros of $G(s)H(s)$ to the right of the point must be **even**. The following example illustrates the determination of the properties of the root loci on the real axis of the s-plane.

▶ EXAMPLE E-6-1 The root loci on the real axis for two pole-zero configurations of $G(s)H(s)$ are shown in Fig. E-6. Notice that the entire real axis is occupied by the root loci for all values of K.

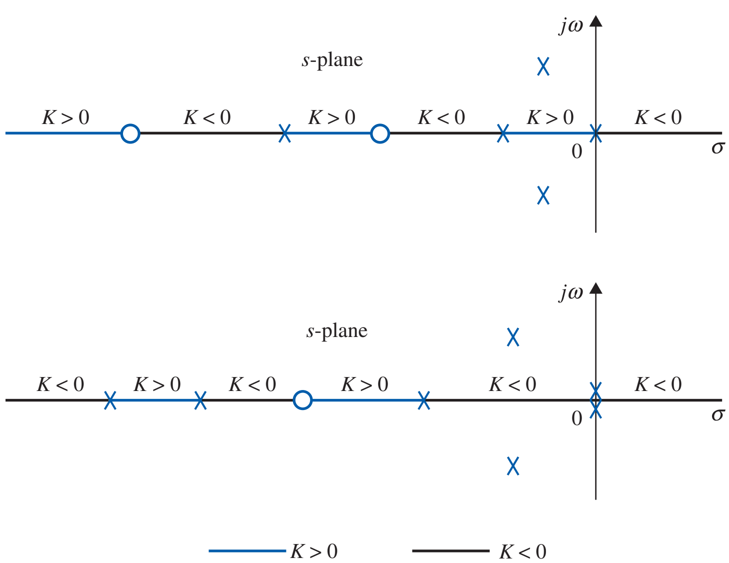

Figure E-6 Properties of root loci on the real axis. ◀

▶ E-7 ANGLES OF DEPARTURE AND ANGLES OF ARRIVAL OF THE ROOT LOCI

The angle of departure or arrival of a root locus at a pole or zero of G(s)H(s) denotes the angle of the tangent to the locus near the point.

The angles of departure and arrival are determined using Eq. (7-18) for root loci for positive K and Eq. (7-19) for root loci for negative K. The details are illustrated by the following example.

▶ EXAMPLE E-7-1 For the root-locus diagram shown in Fig. E-7, the root locus near the pole $s = -1 + j$ may be more accurately sketched by knowing the angle at which the root locus leaves the pole. As shown in Fig. E-7, the angle of departure of the root locus at $s = -1 + j$ is represented by θ_2, measured with respect to the real axis. Let us assign s_1 to be a point on the RL leaving the pole at $-1 + j$ and is very close to the pole. Then, s_1 must satisfy Eq. (7-18). Thus,

$$\angle G(s_1)H(s_1) = -(\theta_1 + \theta_2 + \theta_3 + \theta_4) = (2i + 1)180^\circ \tag{E-16}$$

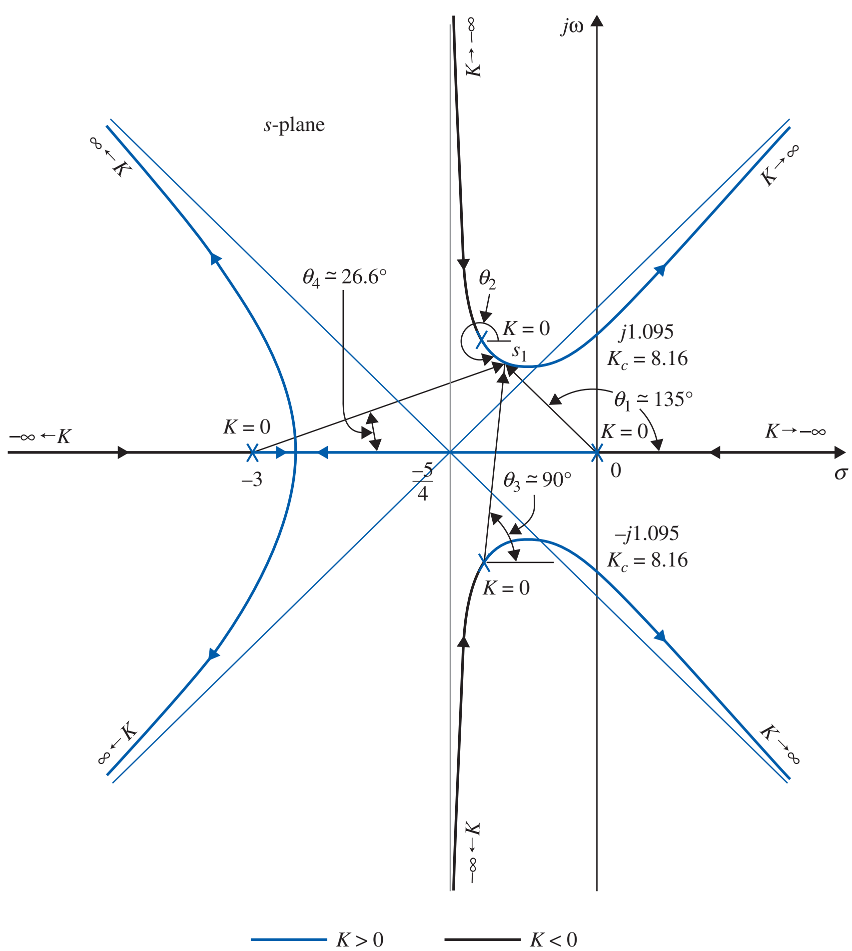

Figure E-7 Root loci of $s(s + 3)(s^2 + 2s + 2) + K = 0$ to illustrate the angles of departure or arrival.

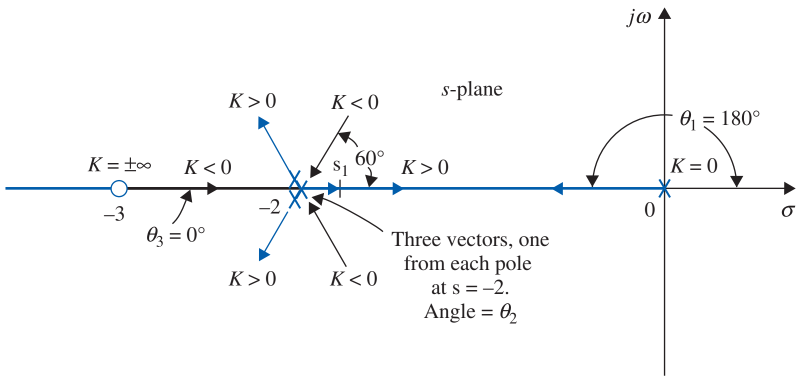

Figure E-8 Angles of departure and arrival at a third-order pole.

where i is any integer. Since s_1 is assumed to be very close to the pole at $-1+j$, the angles of the vectors drawn from the other three poles are approximated by considering that s_1 is at $-1+j$. From Fig. E-7, Eq. (E-16) is written

$$-(135^\circ + \theta_2 + 90^\circ + 26.6^\circ) = (2i+1)180^\circ \tag{E-17}$$

where θ_2 is the only unknown angle. In this case, we can set i to be -1, and the result for θ_2 is -71.6°.

When the angle of departure or arrival of a root locus for positive K at a simple pole or zero of $G(s)H(s)$ is determined, the angle of arrival or departure of the root locus for negative K at the same point differs from this angle by 180°, and Eq. (7-19) is now used. Figure E-7 shows that the angle of arrival of the root locus for negative K at $-1+j$ is 108.4°, which is $180^\circ - 71.6^\circ$. Similarly, for the root-locus diagram in Fig. E-8, we can show that the root locus for negative K arrives at the pole $s=-3$ with an angle of 180°, and the root locus for positive K leaves the same pole at 0°. For the pole at $s=0$, the angle of arrival of the negative-K root locus is 180°, whereas the angle of departure of the positive-K root locus is 180°. These angles are also determined from the knowledge of the type of root loci on sections of the real axis separated by the poles and zeros of $G(s)H(s)$. Since the total angles of the vectors drawn from complex-conjugate poles and zeros to any point on the real axis add up to be zero, the angles of arrival and departure of root loci on the real axis are not affected by the complex poles and zeros of $G(s)H(s)$. ◀

▶ **EXAMPLE E-7-2** In this example we examine the angles of departure and arrival of the root loci at a multiple-order pole or zero of $G(s)H(s)$. Consider that a $G(s)H(s)$ has a multiple-order (third-order) pole on the real axis, as shown in Fig. E-8. Only the real poles and zeros of $G(s)H(s)$ are shown, since the complex ones do not affect the type or the angles of arrival and departure of the root loci on the real axis. For the third-order pole at $s=-2$, there are three positive-K loci leaving and three negative-K loci arriving at the point. To find the angles of departure of the positive-K root loci, we assign a point s_1 on one of the loci near $s=-2$, and apply Eq. (7-18). The result is

$$-\theta_1 - 3\theta_2 + \theta_3 = (2i+1)180^\circ \tag{E-18}$$

where θ_1 and θ_3 denote the angles of the vectors drawn from the pole at 0 and the zero at -3, respectively, to s_1. The angle θ_2 is multiplied by 3, since there are three poles at $s=-2$, so that there are three vectors drawn from -2 to s_1. Setting i to zero in Eq. (E-18), and since $\theta_1 = 180^\circ$, $\theta_3 = 0^\circ$ we have $\theta_2 = 0^\circ$, which is the angle of departure of the positive-K root loci that lies between $s=0$ and $s=-2$. For the angles of departure of the other two positive-K loci, we set $i=1$ and $i=2$ successively in Eq. (E-18), and we have $\theta_2 = 120^\circ$ and -120°. Similarly, for the three negative-K root loci that arrive at $s=-2$, Eq. (7-19) is used, and the angles of arrivals are found to be 60°, 180°, and -60°. ◀

▶ E-8 INTERSECTION OF THE ROOT LOCI WITH THE IMAGINARY AXIS

• Routh-Hurwitz criterion may be used to find the intersection of the root loci on the imaginary axis.

The points where the root loci intersect the imaginary axis of the s-plane, and the corresponding values of K, may be determined by means of the Routh-Hurwitz criterion. For complex situations, when the root loci have multiple number of intersections on the imaginary axis, the intersects and the critical values of K can be determined with the help of the root-locus computer program. The Bode diagram method in Chapters 2 and 8 associated with the frequency response, can also be used for this purpose.

▶ EXAMPLE E-8-1 The root loci shown in Fig. E-7 is for the equation

$$s(s+3)\left(s^2+2s+2\right)+K=0 \tag{E-19}$$

Fig. E-7 shows that the root loci intersect the $j\omega$ axis at two points. Applying the Routh-Hurwitz criterion to Eq. (E-19), and by solving the auxiliary equation, we have the critical value of K for stability at $K = 8.16$, and the corresponding crossover points on the $j\omega$-axis are at $\pm j1.095$. ◀

▶ E-9 BREAKAWAY POINTS

E-9-1 (Saddle Points) on the Root Loci

Breakaway points on the root loci of an equation correspond to multiple-order roots of the equation.

Fig. E-9(a) illustrates a case in which two branches of the root loci meet at the breakaway point on the real axis and then depart from the axis in opposite directions. In this

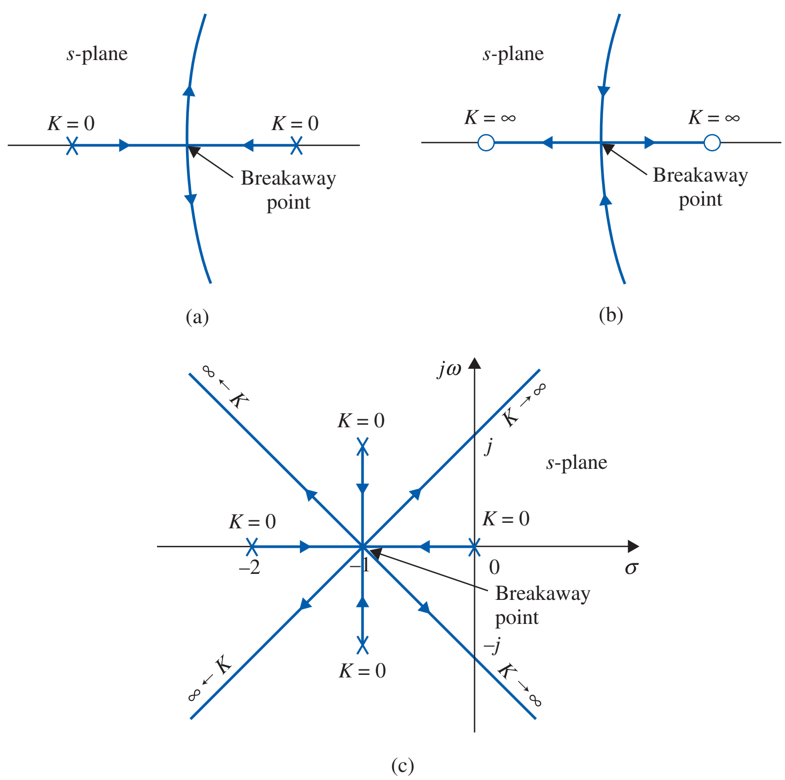

Figure E-9 Examples of breakaway points on the real axis in the s-plane.

case, the breakaway point represents a double root of the equation when the value of K is assigned the value corresponding to the point. Fig. E-9(b) shows another common situation when two complex-conjugate root loci approach the real axis, meet at the breakaway point, and then depart in opposite directions along the real axis. In general, a breakaway point may involve more than two root loci. Fig. E-9(c) illustrates a situation when the breakaway point represents a fourth-order root.

• A root-locus plot may have more than one breakaway points.

• Breakaway points may be complex conjugates in the s-plane

A root-locus diagram can have, of course, more than one breakaway point. Moreover, the breakaway points need not always be on the real axis. Because of the conjugate symmetry of the root loci, the breakaway points not on the real axis must be in complex-conjugate pairs. Refer to Fig. E-12 for an example of root loci with complex breakaway points. The properties of the breakaway points of root loci are given as follows:

The breakaway points on the root loci of $\mathbf{1 + KG_1(s)H_1(s) = 0}$ ***must satisfy***

$$\frac{dG_1(s)H_1(s)}{ds} = 0 \tag{E-20}$$

It is important to point out that the condition for the breakaway point given in Eq. (E-20) is *necessary* but *not sufficient*. In other words, all breakaway points on the root loci for all values of K must satisfy Eq. (E-20), but not all solutions of Eq. (E-20) are breakaway points. To be a breakaway point, the solution of Eq. (E-20) must also satisfy the equation $1 + KG_1(s)H_1(s) = 0$, that is, must also be a point on the root loci for some real K. In general, the following conclusions may be made with regard to the solutions of Eq. (E-20):

1. All *real* solutions of Eq. (E-20) are breakaway points on the root loci for all values of K, since the entire real axis of the s-plane is occupied by the root loci.
2. The complex-conjugate solutions of Eq. (E-20) are breakaway points only if they satisfy the characteristic equation or are points on the root loci.
3. Since the condition of the root loci is

$$K = -\frac{1}{G_1(s)H_1(s)} \tag{E-21}$$

taking the derivative on both sides of the equation with respect to s, we have

$$\frac{dK}{ds} = \frac{dG_1(s)H_1(s)/ds}{[G_1(s)H_1(s)]^2} \tag{E-22}$$

Thus, the breakaway point condition can also be written as

$$\frac{dK}{ds} = 0 \tag{E-23}$$

where K is expressed as in Eq. (E-21).

E-9-2 The Angle of Arrival and Departure of Root Loci at the Breakaway Point

The angles at which the root loci arrive or depart from a breakaway point depend on the number of loci that are involved at the point. For example, the root loci shown in

Figs. E-9(a) and E-9(b) all arrive and break away at 180° apart, whereas in Fig. E-9(c), the four root loci arrive and depart with angles 90° apart, whereas in Fig. E-9(c), the four root loci arrive and depart with angles 90° apart. In general, ***n root loci*** **(**$-\infty \le K \le \infty$**)** ***arrive or depart a breakaway point at 180/n degrees apart***.

Many root-locus computer programs have features that will obtain the breakaway points, which a rather tedious task to do manually.

▶ EXAMPLE E-9-1 Consider the second-order equation

$$s(s+2) + K(s+4) = 0 \tag{E-24}$$

Based on some of the properties of the root loci described thus far, the root loci of Eq. (E-24) are sketched as shown in Fig. E-10 for $-\infty < K < \infty$. It can be proven that the complex portion of the root loci is a circle. The two breakaway points are on the real axis, one between 0 and -2 and the other between -4 and $-\infty$. From Eq. (E-24), we have

$$G_1(s)H_1(s) = \frac{s+4}{s(s+2)} \tag{E-25}$$

Applying Eq. (E-20), the breakaway points on the root loci must satisfy

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{s(s+2) - 2(s+1)(s+4)}{s^2(s+2)^2} = 0 \tag{E-26}$$

or

$$s^2 + 8s + 8 = 0 \tag{E-27}$$

Solving Eq. (E-27), we find the two breakaway points of the root loci at $s = -1.172$ and -6.828. Fig. E-10 shows that the two breakaway points are all on the root loci for positive K.

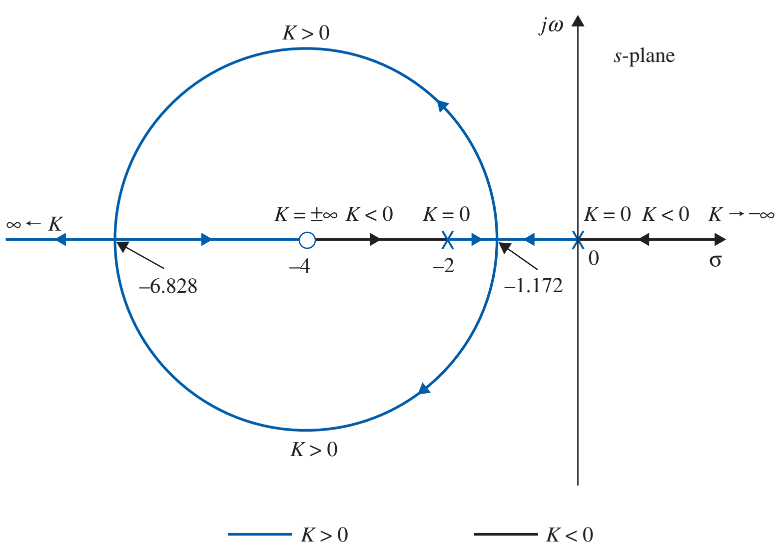

Figure E-10 Root loci of $s(s + 2) + K(s + 4) = 0$. ◀

▶ EXAMPLE E-9-2 Consider the equation

$$s^2 + 2s + 2 + K(s+2) = 0 \tag{E-28}$$

The equivalent $G(s)H(s)$ is obtained by dividing both sides of Eq. (E-28) by the terms that do not contain K. We have

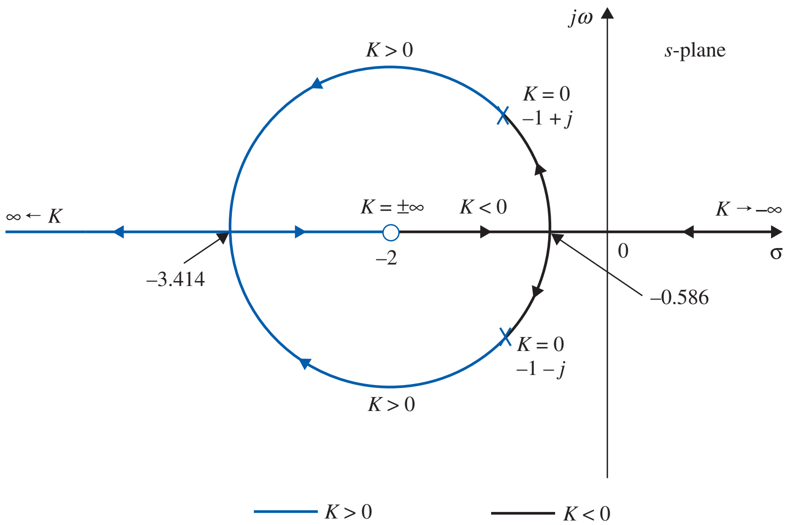

Figure E-11 Root loci of $s^2 + 2s + 2 + K(s + 2) = 0$.

$$G(s)H(s) = \frac{K(s+2)}{s^2+2s+2} \tag{E-29}$$

Based on the poles and zeros of $G(s)H(s)$, the root loci of Eq. (E-29) are plotted as shown in Fig. E-11. The plot shows that there are two breakaway points, one for $K > 0$ and one for $K < 0$. These breakaway points are determined from

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{d}{ds}\left(\frac{s+2}{s^2+2s+2}\right) = \frac{s^2+2s+2-2(s+1)(s+2)}{(s^2+2s+2)^2} = 0 \tag{E-30}$$

or

$$s^2 + 4s + 2 = 0 \tag{E-31}$$

The solution of this equation gives the breakaway point as $s = -0.586$ and $s = -3.414$. ◀

▶ **EXAMPLE E-9-3** Fig. E-12 shows the root loci of the equation

$$s(s+4)(s^2+4s+20) + K = 0 \tag{E-32}$$

Dividing both sides of the last equation by the terms that do not contain K, we have

$$1 + KG_1(s)H_1(s) = 1 + \frac{K}{s(s+4)(s^2+4s+20)} = 0 \tag{E-33}$$

Since the poles of $G_1(s)H_1(s)$ are symmetrical about the axes $\sigma = -2$ *and* $\omega = 0$ *in the* s-plane, the root loci of the equation are also symmetrical with respect to these two axes. Taking the derivative of $G_1(s)H_1(s)$ with respect to s, we get

$$\frac{dG_1(s)H_1(s)}{ds} = -\frac{4s^3+24s^2+72s+80}{[s(s+4)(s^2+4s+20)]^2} = 0 \tag{E-34}$$

or

$$s^3 + 6s^2 + 18s + 20 = 0 \tag{E-35}$$

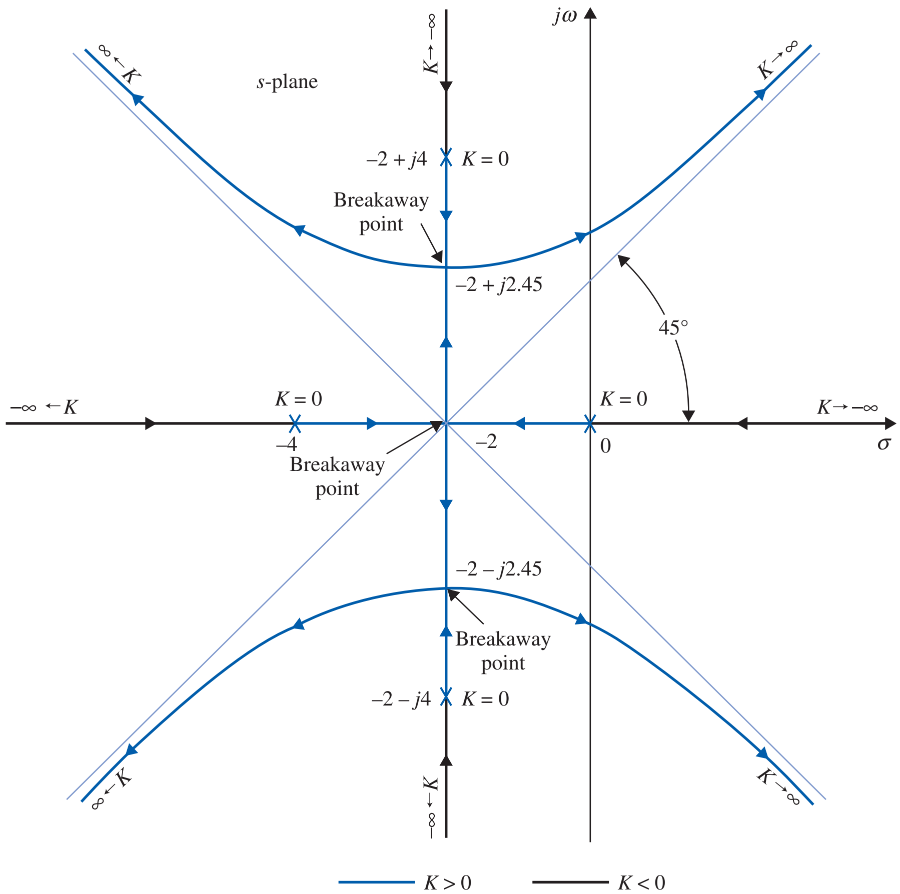

Figure E-12 Root loci of $s(s+4)(s^2+4s+20)+K=0$.

The solutions of the last equation are $s = -2$, $-2 + j2.45$, and $-2 - j2.45$. In this case, Fig. E-12 shows that all the solutions of Eq. (E-35) are breakaway points on the root loci, and two of these points are complex. ◀

▶ **EXAMPLE E-9-4** In this example, we shall show that not all the solutions of Eq. (E-20) are breakaway points on the root loci. The root loci of the equation

$$s\left(s^2+2s+2\right)+K=0 \tag{E-36}$$

are shown in Fig. E-13. The root loci show that neither the $K \geq 0$ loci nor the $K \leq 0$ loci has any breakaway point in this case. However, writing Eq. (E-36) as

$$1+KG_1(s)H_1(s)=1+\frac{K}{s(s^2+2s+2)}=0 \tag{E-37}$$

and applying Eq. (E-20), we have the equation for the breakaway points:

$$3s^2+4s+2=0 \tag{E-38}$$

The roots of Eq. (E-38) are $s = -0.667 + j0.471$ and $-0.667 - j0.471$. These two roots are **not** breakaway points on the root loci, since they do not satisfy Eq. (E-36) for any real value of K.

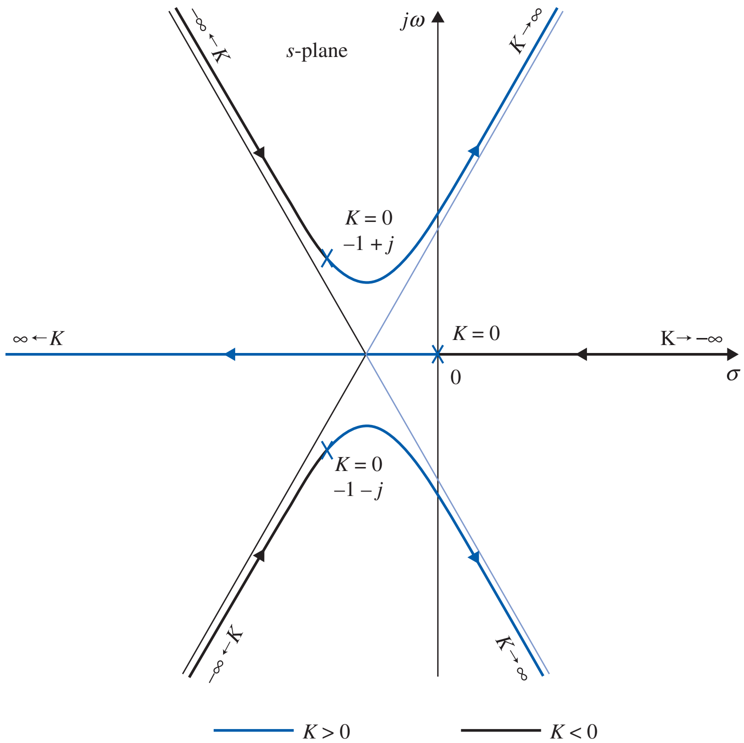

Figure E-13 Root loci of $s(s^2 + 2s + 2) + K = 0$.

▶ E-10 CALCULATION OF *K* ON THE ROOT LOCI

Once the root loci are constructed, the values of K at any point s_1 on the loci can be determined by use of the defining equation of Eq. (7-20). Graphically, the magnitude of K can be written as

$$|K| = \frac{\Pi \text{ lengths of vectors drawn from the poles of } G_1(s)H_1(s) \text{ to } s_1}{\Pi \text{ lengths of vectors drawn from the zeros of } G_1(s)H_1(s) \text{ to } s_1} \tag{E-39}$$

▶ **EXAMPLE E-10-1** As an illustration on the determination of the value of K on the root loci, the root loci of the equation

$$s^2 + 2s + 2 + K(s + 2) = 0 \tag{E-40}$$

are shown in Fig. E-14. The value of K at the point s_1 is given by

$$K = \frac{A \times B}{C} \tag{E-41}$$

where A and B are the lengths of the vectors drawn from the poles of $G(s)H(s) = K(s+2)/(s^2 + 2s + 2)$ to the point s_1, and C is the length of the vector drawn from the zero of $G(s)H(s)$ to s_1. In this case, s_1 is on the locus where K is positive. In general, the value of K at the point where the root loci intersect the imaginary axis can also be found by the method just described. Figure E-14 shows that the value of K at $s = 0$ is -1. The computer method and the Routh-Hurwitz criterion are other convenient alternatives of finding the critical value of K for stability.

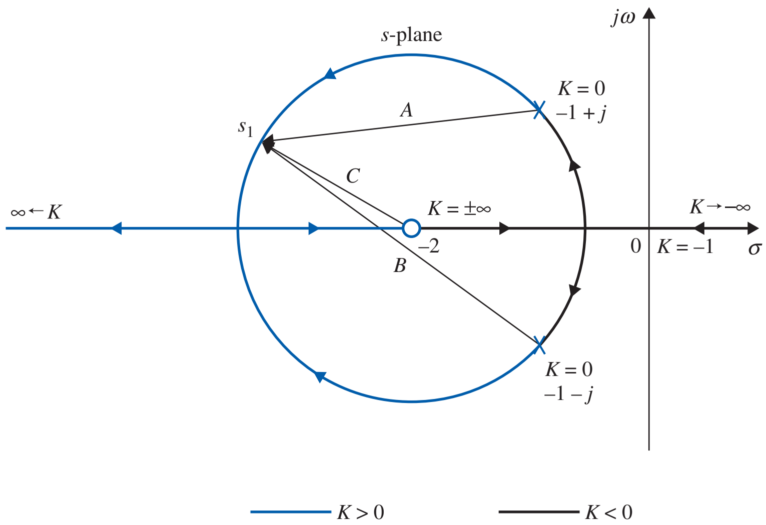

Figure E-14 Graphical method of finding the values of K on the real axis.

In summary, except for extremely complex cases, the properties on the root loci presented here should be adequate for making a reasonably accurate sketch of the root-locus diagram short of plotting it point by point. The computer program can be used to solve for the exact root locations, the breakaway points, and some of the other specific details of the root loci, including the plotting of the final loci. However, one cannot rely on the computer solution completely, since the user still has to decide on the range and resolution of K so that the root-locus plot has a reasonable appearance. For quick reference, the important properties described are summarized in Table E-1.

TABLE E-1 Properties of the Root Loci of $F(s) = 1 + KG_1(s)H_1(s) = 0$

1. $K = 0$ points	The $K = 0$ points are at the poles of $G(s)H(s)$, including those at $s = \infty$.
2. $K = \pm\infty$ points	The $K = \pm\infty$ points are at the zeros of $G(s)H(s)$, including those at $s = \infty$.
3. Number of separate root loci	The total number of root loci is equal to the order of the characteristic equation $F(s)$
4. Symmetry of root loci	The root loci are symmetrical about the axes of symmetry of the pole-zero configuration of $G(s)H(s)$.
5. Asymptotes of root loci as $\|s\| \to \infty$	For large values of s, the root loci for $K > 0$ are asymptotic to asymptotes with angles given by $\theta_i = \frac{2i+1}{\|n-m\|} \times 180°$ For $K < 0$, the root loci are asymptotic to $\theta_i = \frac{2i}{\|n-m\|} \times 180°$ where $i = 0, 1, 2, \ldots, \|n-m\| - 1$, n = number of finite poles of $G(s)H(s)$, and m = number of finite zeros of $G(s)H(s)$.

(Continued)

TABLE E-1 (Continued)

6.	Intersection of the asymptotes	(a) The intersection of the asymptotes lies only on the real axis in the s-plane. (b) The point of intersection of the asymptotes is given by

$$\sigma_1 = \frac{\Sigma \text{ real parts of poles of } G(s)H(s) - \Sigma \text{ real parts of zeros of } G(s)H(s)}{n-m}$$

7.	Root loci on the real axis	Root loci for $K > 0$ are found in a section of the real axis only if the total number of real poles and zeros of $G(s)H(s)$ to the **right** of the section is **odd**. If the total number of real poles and zeros to the right of a given section is **even**, root loci for $K < 0$ are found.
8.	Angles of departure	The angle of departure or arrival of the root loci from a pole or a zero of $G(s)H(s)$ can be determined by assuming a point s_1 that is very close to the pole, or zero, and applying the equation $\angle G(s_1)H(s_1) = \sum_{k=1}^{m} \angle(s_1 + z_k) - \sum_{j=1}^{n} \angle(s_1 + p_j)$ $= 2(i+1)180^\circ \quad K > 0$ $= 2i \times 180^\circ \quad K < 0$ where $i = 0, \pm 1, \pm 2, \ldots$
9.	Intersection of the root loci	The crossing points of the root loci on the imaginary axis and with the imaginary axis the corresponding values of K may be found by use of the Routh-Hurwitz criterion.
10.	Breakaway points	The breakaway points on the root loci are determined by finding the roots of $dK/ds = 0$, or $dG(s)H(s)/ds = 0$. These are necessary conditions only.
11.	Calculation of the values of K	The absolute value of K at any point s_1 on the root loci is on the root loci determined from the equation $\lvert K \rvert = \frac{1}{\lvert G_1(s_1)H_1(s_1) \rvert}$

The following example illustrates the construction of a root locus diagram manually, step by step, using the root locus properties given in Table E-1.

▶ EXAMPLE E-10-2 Consider the equation

$$s(s+5)(s+6)\left(s^2+2s+2\right) + K(s+3) = 0 \tag{E-42}$$

Dividing both sides of the last equation by the terms that do not contain K, we have

$$G(s)H(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)} \tag{E-43}$$

The following properties of the root loci are determined:

1. The $K = 0$ points are at the poles of $G(s)H(s)$: $s = -5, -6, -1 + j$, and $-1 - j$.
2. The $K = \pm\infty$ points are at the zeros of $G(s)H(s)$: $s = -3, \infty, \infty, \infty, \infty$.

3. There are five separate branches on the root loci.
4. The root loci are symmetrical with respect to the real axis of the s-plane.
5. Since $G(s)H(s)$ has five poles and one finite zero, four RL and CRL should approach infinity along the asymptotes. The angles of the asymptotes of the RL are given by [Eq. (E-9)]

$$\theta_i = \frac{2i+1}{|n-m|}180^\circ = \frac{2i+1}{|5-1|}180^\circ \quad 0 \le K < \infty \tag{E-44}$$

for $i = 0, 1, 2, 3$. Thus, the four root loci that approach infinity as K approaches infinity should approach asymptotes with angles of 45°, −45°, 135°, and −135°, respectively. The angles of the asymptotes of the CRL at infinity are given by Eq. (E-10):

$$\theta_i = \frac{2i}{|n-m|}180^\circ = \frac{2i}{|5-1|}180^\circ \quad -\infty < K \le 0 \tag{E-45}$$

for $i = 0, 1, 2, 3$. Thus, as K approaches $-\infty$, four root loci for $K < 0$ should approach infinity along asymptotes with angles of 0°, 90°, 180°, and 270°.

6. The intersection of the asymptotes is given by [Eq. (E-12)]

$$\sigma_1 = \frac{\Sigma(-5-6-1-1)-(-3)}{4} = -2.5 \tag{E-46}$$

The results from these six steps are illustrated in Fig. E-15. It should be pointed out that in general the properties of the asymptotes do not indicate on which side of the asymptotes the root loci lie. The asymptotes indicate nothing more than the behavior of the root loci as

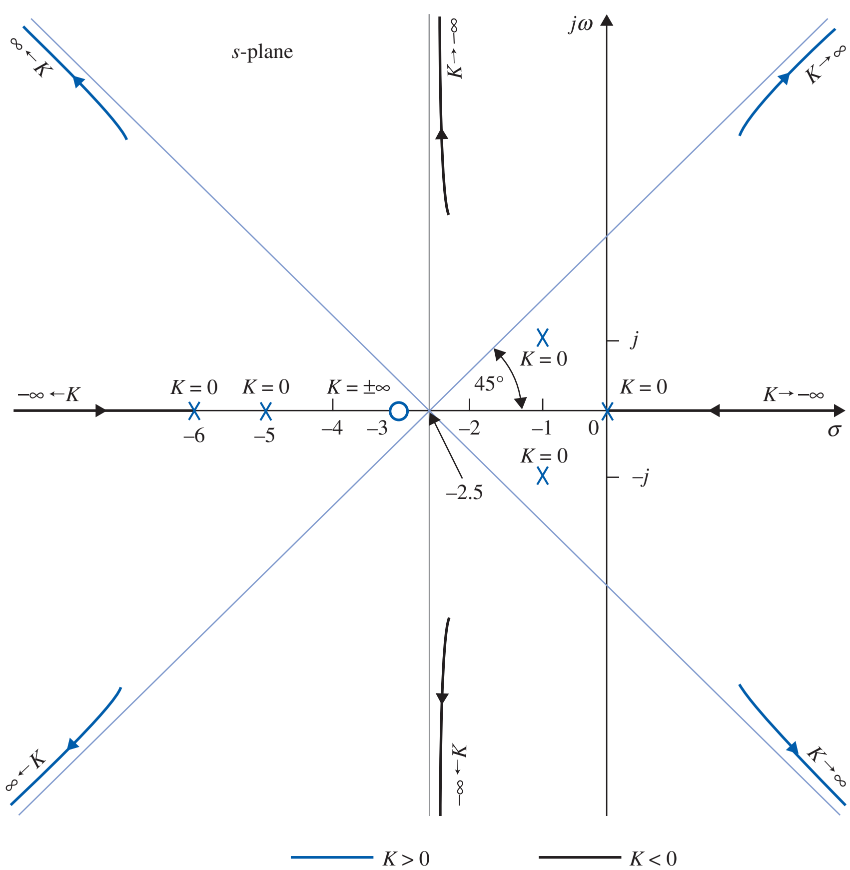

Figure E-15 Preliminary calculation of the root loci of $s(s+5)(s+6)(s^2+2s+2) + K(s+3) = 0$.

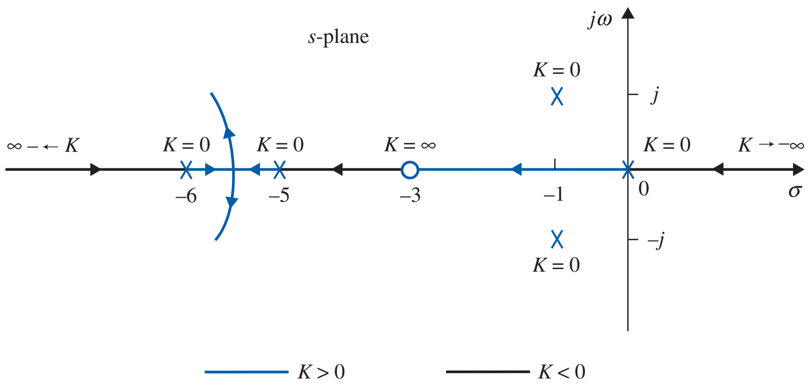

Figure E-16 Root loci of $s(s+5)(s+6)(s^2+2s+2)+K(s+3)=0$ on the real axis.

$|s| \rightarrow \infty$ In fact, the root locus can even cross an asymptote in the finite s domain. The segments of the root loci shown in Fig. E-15 can be accurately plotted only if additional information is obtained.

7. Root loci on the real axis: There are $K \geq 0$ root loci on the real axis between $s=0$ and -3, and $s=-5$ and -6. There are $K \leq 0$ root loci on the remaining portions of the real axis, that is, between $s=-3$ and -5, and $s=-6$ and $-\infty$, as shown in Fig. E-16.
8. Angles of departure: The angle of departure θ of the root loci leaving the pole at $-1+j$ is determined using Eq. (7-18). If s_1 is a point on the root loci leaving the pole at $-1+j$, and s_1 is very close to the pole, as shown in Fig E-17, Eq. (7-18) gives

$$\begin{aligned}\angle(s_1+3)-\angle s_1-\angle(s_1+1+j)-\angle(s_1+5)-\angle(s_1+1-j)\\ =(2i+1)180^\circ\end{aligned} \tag{E-47}$$

or

$$26.6^\circ-135^\circ-90^\circ-14^\circ-11.4^\circ-\theta \cong (2i+1)180^\circ \tag{E-48}$$

for $i=0, \pm 1, \pm 2, \ldots$ Therefore, selecting $i=2$,

$$\theta \cong -43.8^\circ \tag{E-49}$$

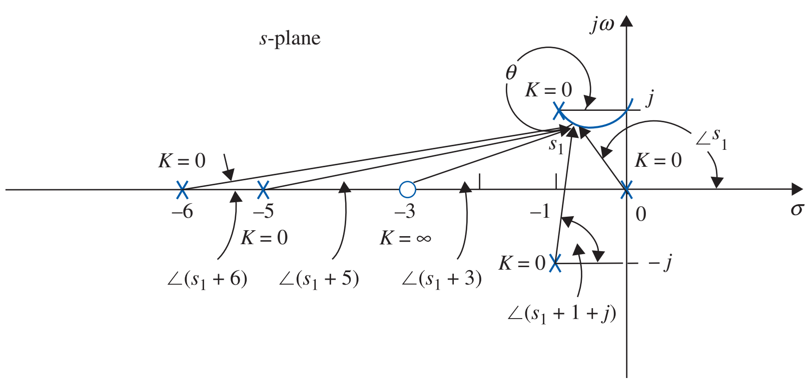

Figure E-17 Computation of angle of departure of the root loci of $s(s+5)(s+6)(s^2+2s+2)+K(s+3)=0$.

Similarly, Eq. (7-19) is used to determine the angle of arrival θ' of the $K \leq 0$ root loci arriving at the pole $-1 + j$. It is easy to see that θ' differs from θ by 180°; thus,

$$\theta' = 180° - 43.8° = 136.2° \tag{E-50}$$

9. The intersection of the root loci on the imaginary axis is determined using Routh's tabulation. Equation (E-42) is written

$$s^5 + 13s^4 + 54s^3 + 82s^2 + (60 + K)s + 3K = 0 \tag{E-51}$$

Routh's tabulation is

s^5	1	54	$60 + K$
s^4	13	82	$3K$
s^3	47.7	$0.769K$	0
s^2	$65.6 - 0.212K$	$3K$	0
s^1	$\dfrac{3940 - 105K - 0.163K^2}{65.6 - 0.212K}$	0	0
s^0	$3K$	0	0

For Eq. (E-51) to have no roots on the imaginary axis or in the right-half of the s-plane, the elements in the first column of Routh's tabulation must all be of the same sign. Thus, the following inequalities must be satisfied:

$$65.6 - 0.212K > 0 \quad \text{or} \quad K < 309 \tag{E-52}$$

$$3940 - 105K - 0.163K^2 > 0 \quad \text{or} \quad K < 35 \tag{E-53}$$

$$K > 0 \tag{E-54}$$

Thus, all the roots of Eq. (E-51) will stay in the left-half s-plane if K lies between 0 and 35, which means that the root loci of Eq. (E-51) cross the imaginary axis when $K = 35$ and $K = 0$. The coordinates at the crossover points on the imaginary axis that correspond to $K = 35$ are determined from the auxiliary equation:

$$A(s) = (65.6 - 0.212K)s^2 + 3K = 0 \tag{E-55}$$

which is obtained by using the coefficients from the row just above the row of zeros in the s^1 row that would have happened when K is set to 35. Substituting $K = 35$ in Eq. (E-55), we get

$$58.2s^2 + 105 = 0 \tag{E-56}$$

The roots of Eq. (E-56) are $s = j1.34$ and $-j1.34$, which are the points at which the root loci cross the $j\omega$-*axis.*

10. Breakaway points: Based on the information gathered from the preceding nine steps, a trial sketch of the root loci indicates that there can be only one breakaway point on the entire root loci, and the point should lie between the two poles of $G(s)H(s)$ at $s = -5$ and -6. To find the breakaway point, we take the derivative on both sides of Eq. (E-43) with respect to s and set it to zero; the resulting equation is

$$s^5 + 13.5s^4 + 66s^3 + 142s^2 + 123s + 45 = 0 \tag{E-57}$$

Since there is only one breakaway expected, only one root of the last equation is the correct solution of the breakaway point. The five roots of Eq. (E-57) are:

$$s = 3.33 + j1.204 \qquad s = 3.33 - j1.204$$
$$s = -0.656 + j0.468 \qquad s = -0.656 - j0.468$$
$$s = -5.53$$

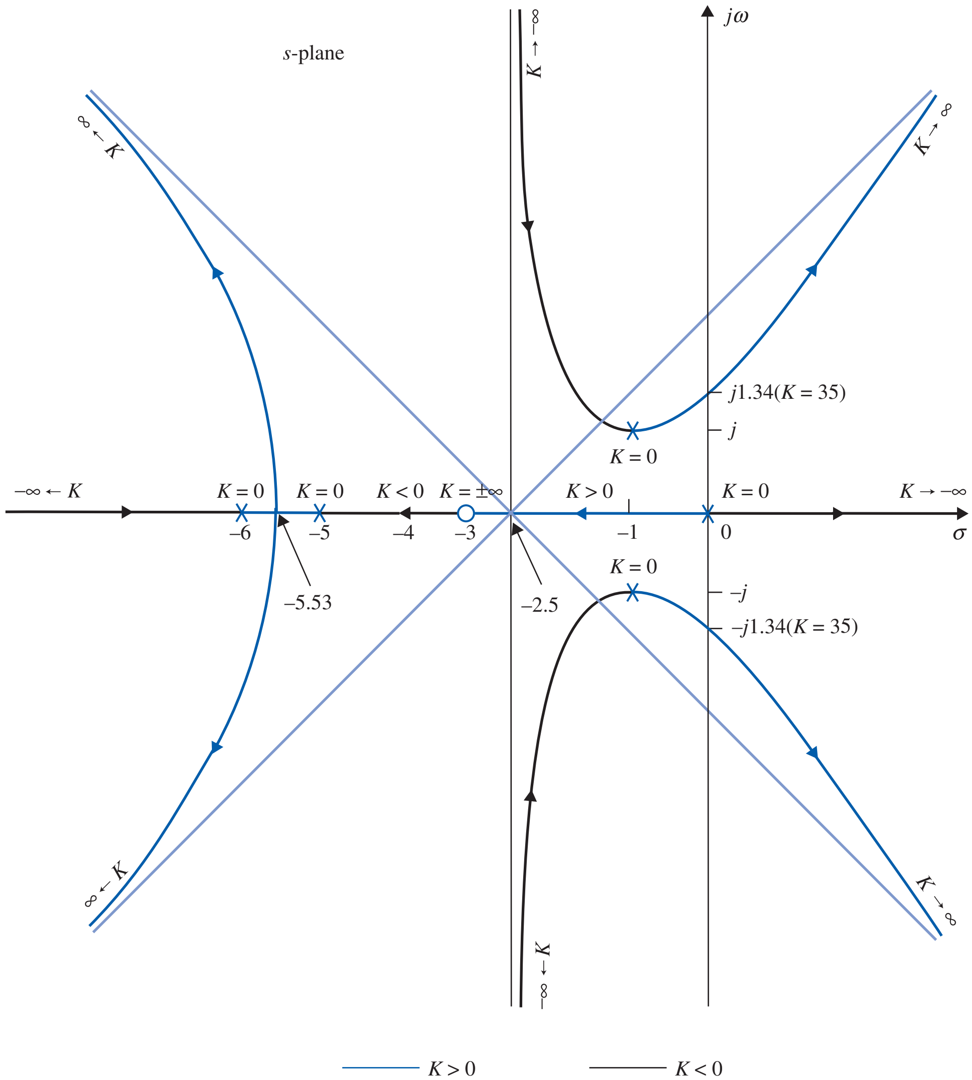

Figure E-18 Root loci of $s(s+5)(s+6)(s^2+2s+2)+K(s+3)=0$.

Clearly, the breakaway point is at -5.53. The other four solutions do not satisfy Eq. (E-51) and are not breakaway points. Based on the information obtained in the last 10 steps, the root loci of Eq. (E-51) are sketched as shown in Fig. E-18. ◀