APPENDIX **F**

# General Nyquist Criterion

## F-1 FORMULATION OF NYQUIST CRITERION

The original Nyquist criterion presented in Chapter 8 is cumbersome to apply when the loop transfer function is of the nonminimum-phase type; that is, $L(s)$ has either poles and/or zeros in the right-half $s$-plane. We shall show that if the loop transfer function is of the nonminimum-phase type, then plotting the Nyquist plot of $L(s)$ only for $s = j\omega$ to $s = j0$ and not enclosing the $(-1, j0)$ point in the $L(s)$-plane is only a necessary, but not sufficient, condition for closed-loop stability. For a system with a nonminimum-phase loop transfer function $L(s)$, the original Nyquist criterion requires that the $L(s)$ plot that corresponds to the entire Nyquist path in Fig. 8-20 be made. If the loop transfer function $L(s)$ has poles or zeros on the $j\omega$-axis, then the Nyquist path of Fig. 8-20 must have small indentations around them on the $j\omega$-axis. This adds even more complexity to the construction of the $L(s)$ plot. Our MATLAB Toolbox (**ACSYS**) or other computer software can all be used to construct the plots of only functions that correspond to the positive $j\omega$-axis of the $s$-plane. The rest of the Nyquist plot that corresponds to the small indentations and the large semicircle on the Nyquist path have to be plotted manually. With modern computer facilities and software, the analyst should not be burdened with the chores of manual plotting. Therefore, we are introducing *a simplified Nyquist criterion that can be applied by using only the positive* j$\omega$*-axis of the Nyquist path and then observing its behavior with reference to the* (−1, j0) *point.*

Yeung [1] introduced a general and yet simplified version of the Nyquist criterion that allows the determination of stability of closed-loop systems of minimum- as well as nonminimum-phase loop transfer functions by using only the positive part of the $j\omega$-axis of the Nyquist path. However, if the system is of the minimum-phase type, the test of whether the $(-1, j0)$ point is enclosed is still simpler to apply. We shall show that, for nonminimum-phase systems, if the $(-1, j0)$ point is enclosed, the system is still unstable. However, if the $(-1, j0)$ point is not enclosed, then an additional angle condition is all that must be satisfied by the Nyquist plot of $L(s)$ for the system to be stable.

Let us consider the two Nyquist paths shown in Figs. F-1(a) and F-1(b). Apparently, the Nyquist path $\Gamma_{s1}$ in Fig. F-1(a) is the original one shown in Fig. 8-20, whereas the path $\Gamma_{s2}$ in Fig. F-1(b) encircles not only the entire right-half $s$-plane but also all the poles and zeros of $L(s)$ on the $j\omega$-axis, if there are any. Let us define the following quantities.

$Z$ = number of zeros of $1 + L(s)$ that are in the right-half $s$-plane.

$P$ = number of poles of $L(s)$, or of $1 + L(s)$, that are in the right-half $s$-plane.

$P_\omega$ = number of poles of $L(s)$, or of $1 + L(s)$, that are on the $j\omega$-axis, including the origin.

$N_1$ = number of times the $(-1, j0)$ point of the $L(s)$-plane that is encircled by the Nyquist plot of $L(s)$ corresponding to $\Gamma_{s1}$.

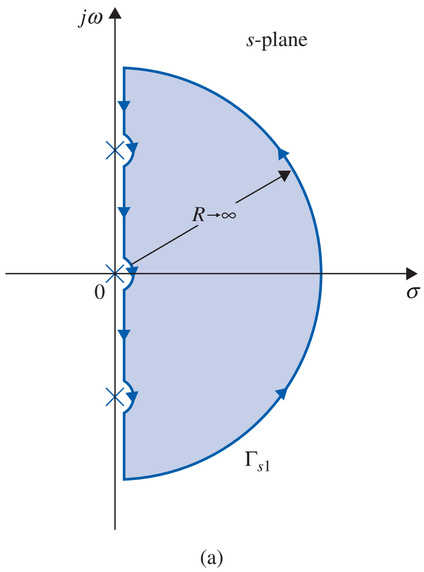


**Figure F-1** (a) Nyquist path. (b) An alternative Nyquist path. (Source: K. S. Yeung, "A Reformulation of Nyquist's Criterion," *IEEE Trans. Educ.*, Vol. E-28, pp. 58–60, Feb. 1985.)

$N_2$ = number of times the $(-1, j0)$ point of the $L(s)$-plane that is encircled by the Nyquist plot of $L(s)$ corresponding to $\Gamma_{s2}$.

Then, with reference to the two Nyquist paths in Fig. F-1, and according to the Nyquist criterion,

$$N_1 = Z - P \tag{F-1}$$

and

$$N_2 = Z - P - P_\omega \tag{F-2}$$

Let $\Phi_1$ and $\Phi_2$ represent the net angles traversed by the Nyquist plot of $L(s)$ with respect to the $(-1, j0)$ point, corresponding to $\Gamma_{s1}$ and $\Gamma_{s2}$, respectively. Then,

$$\Phi_1 = N_1 \times 360^\circ = (Z - P)360^\circ \tag{F-3}$$

$$\Phi_2 = N_2 \times 360^\circ = (Z - P - P_\omega)360^\circ \tag{F-4}$$

Let us consider that each of the Nyquist paths $\Gamma_{s1}$ and $\Gamma_{s2}$ is composed of three portions:

1. The portion from $s = -j\infty$ to $+j\infty$ along the semicircle with infinite radius
2. The portion along the $j\omega$-axis, excluding all the small indentations
3. All the small indentations on the $j\omega$-axis

Because the Nyquist paths in Fig. F-1 are symmetrical about the real axis in the $s$-plane, the angles traversed by the Nyquist plots are identical for positive and negative values of $\omega$. Thus, $\Phi_1$ and $\Phi_2$ are written

$$\Phi_1 = 2\Phi_{11} + \Phi_{12} + \Phi_{13} \tag{F-5}$$

$$\Phi_2 = 2\Phi_{11} - \Phi_{12} + \Phi_{13} \tag{F-6}$$

where

$\Phi_{11}$ = angle traversed by the Nyquist plot of $L(s)$ with respect to the $(-1, j0)$ point, corresponding to the positive $j\omega$-axis or the $-j\omega$-axis of the $s$-plane, excluding the small indentations.

$\Phi_{12}$ = angle traversed by the Nyquist plot of $L(s)$ with respect to the $(-1, j0)$ point, corresponding to the small indentations on the $j\omega$-axis of $\Gamma_{s1}$. Because on $\Gamma_{s2}$ the directions of the small indentations are opposite to that of $\Gamma_{s1}$, the sign of $\Phi_{12}$ in Eq. (F-5) is negative.

$\Phi_{13}$ = angle traversed by the Nyquist plot of $L(s)$ with respect to the $(-1, j0)$ point, corresponding to the semicircle with infinite radius on the Nyquist paths.

For a transfer function $L(s)$ that does not have more zeros than poles, the Nyquist plot of $L(s)$ that corresponds to the infinite semicircle must either be a point on the real axis or a trajectory around the origin of the $L(s)$-plane. Thus, the angle $\Phi_{13}$ traversed by the phasor drawn from the $(-1, j0)$ point to the Nyquist plot along the semicircle with infinite radius is always zero.

Now adding Eq. (F-5) to Eq. (F-6) and using Eqs. (F-3) and (F-4), we get

$$\begin{aligned}\Phi_1 + \Phi_2 &= 4\Phi_{11} \\ &= (2Z - 2P - P_\omega)360^\circ\end{aligned} \tag{F-7}$$

Solving for $\Phi_{11}$, we get

$$\Phi_{11} = (Z - P - 0.5P_\omega)180^\circ \tag{F-8}$$

The equation states:

The total angle traversed by the phasor drawn from the $(-1, j0)$ point to the $L(s)$ Nyquist plot that corresponds to the portion on the positive $j\omega$-axis of the $s$-plane, excluding the small indentations, if any, equals

The number of zeros of $1 + L(s)$ in the right-half $s$-plane

The number of poles of $L(s)$ in the right-half $s$-plane (F-9)

0.5(the number of poles of $L(s)$ on the $j\omega$ = axis)180°

Thus, *the Nyquist stability criterion can be carried out by constructing only the Nyquist plot that corresponds to the $s = j\infty$ to $s = 0$ portion on the Nyquist path.* Furthermore, if the closed-loop system is unstable, by knowing the values of $\Phi_{11}$, $P_\omega$, and $P$, Eq. (F-8) gives the number of roots of the characteristic equation that are in the right-half $s$-plane.

For the closed-loop system to be stable, $Z$ must equal zero. Thus, the Nyquist criterion for stability of the closed-loop system is

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ \tag{F-10}$$

Because $P_\omega$ and $P$ cannot be negative, the last equation indicates that ***if the phase traversed by the Nyquist plot of $L(j\omega)$ as $\omega$ varies from $\infty$ to 0, $\Phi_{11}$, is positive with respect to the $(-1, j0)$ point, the closed-loop system is unstable.***

However, if $\Phi_{11}$ is negative, it still has to satisfy Eq. (F-9) for the system to be stable. With reference to the Nyquist plot of $L(j\omega)$ and the $(-1, j0)$ point, we see that, when the angle variation $\Phi_{11}$ is positive, it corresponds to the $(-1, j0)$ point being enclosed. Thus, the condition that the Nyquist plot of $L(j\omega)$ not enclosing the $(-1, j0)$ point is a necessary condition for closed-loop stability for nonminimum-phase systems. However, if the $(-1, j0)$ point is not enclosed by the Nyquist plot of $L(j\omega)$, for the nonminimum-phase system to be closed-loop stable, the angle variation $\Phi_{11}$ still has to satisfy Eq. (F-9).

### F-1-1 System with Minimum-Phase Loop Transfer Functions

If $L(s)$ is of the minimum-phase type, then $P = 0$ and $P_\omega$ denotes the number of poles of $L(s)$ that are at the origin. Eq. (F-8) becomes

$$\Phi_{11} = (Z - 0.5P_\omega)180^\circ \tag{F-11}$$

For closed-loop stability, $Z = 0$; Eq. (F-11) becomes

$$\Phi_{11} = -P_\omega \times 90^\circ \tag{F-12}$$

Because $P_\omega$ denotes the number of poles of $L(s)$ that are at the origin, it is easy to see that, if the $(-1, j0)$ point is not enclosed by the Nyquist plot of $L(s)$, $\Phi_{11}$ will always be given by Eq. (F-12). Thus, when $L(s)$ is of the minimum-phase type, the condition that the $(-1, j0)$ point not be enclosed by the Nyquist plot is a necessary and sufficient condition for closed-loop stability.

### F-1-2 Systems with Improper Loop Transfer Functions

Eq. (F-8) is derived based on the condition that $\Phi_{13} = 0$, which is true only if $L(s)$ is strictly proper; that is, it has more poles than zeros. For improper transfer functions, we can again use the method discussed in Section 8-6 by plotting the Nyquist plot of $1/L(s)$.

## ▶ F-2 ILLUSTRATIVE EXAMPLES—GENERAL NYQUIST CRITERION MINIMUM AND NONMINIMUM TRANSFER FUNCTIONS

In the following example we shall show that the Nyquist plot of a nonminimum-phase transfer function does not enclose the $(-1, j0)$ point, and yet the system is unstable.

**▶ EXAMPLE F-2-1** Consider that the loop transfer function of a control system is given by

$$L(s) = \frac{s^2 - s + 1}{s(s^2 - 6s + 5)} \tag{F-13}$$

Because $L(s)$ has one pole at the origin and two poles in the right-half $s$-plane, $P_\omega = 1$ and $P = 2$. From Eq. (F-9), the closed-loop system is stable if the following condition is satisfied:

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = -450^\circ \tag{F-14}$$

The Nyquist plot of $L(j\omega)$ for $\omega = \infty$ to $\omega = 0$ is plotted as shown in Fig. F-2. Apparently, in this case, the $(-1, j0)$ point is not enclosed by the Nyquist plot. However, because $\Phi_{11}$ is $-90^\circ$, and not $-450^\circ$, the system is unstable. Substituting $\Phi_{11} = 90^\circ$ into Eq. (8-82) and solving for $Z$, we have $Z = 2$, which means that there are two closed-loop poles in the right-half $s$-plane. ◀

**▶ EXAMPLE F-2-2** Consider the system described in Example 8-1-1. The loop transfer function of the system, $L(s)$, is given in Eq. (9-59) and is repeated below.

$$L(j\omega) = \frac{K}{s(s+2)(s+10)} \tag{F-15}$$

The Nyquist plot of $L(j\omega)$ is shown in Fig. 8-25. It is shown in Example 8-1-1 that, for closed-loop stability, the $(-1, j0)$ point in the $L(j\omega)$-plane must be to the left of the intersect of the Nyquist plot with the real axis.

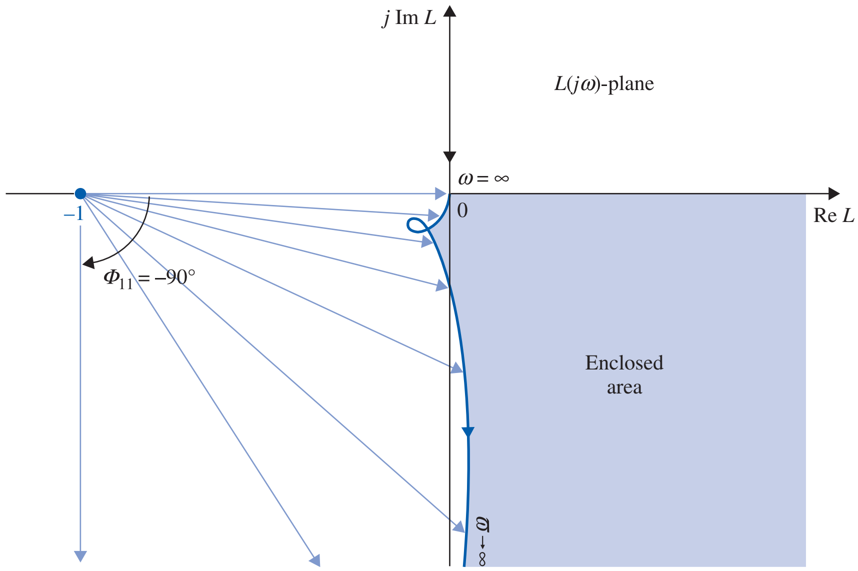


**Figure F-2** Nyquist plot of $L(s) = \dfrac{s^2 - s + 1}{s(s^2 - 6s + 5)}$.

Now we shall apply the generalized Nyquist criterion to the system. Because $L(s)$ is of the minimum-phase type, $P = 0$, and it has one pole at the origin, thus, $P_\omega = 1$. Substituting these quantities into Eq. (F-11), we have

$$\Phi_{11} = (Z - 0.5)180^\circ \tag{F-16}$$

For closed-loop stability, $Z$ must equal zero; thus, the last equation gives $\Phi_{11} = -90^\circ$. This means that the phasor drawn from the $(-1, j0)$ point to the Nyquist plot, from $\omega = \infty$ to $\omega = 0$, must equal $-90^\circ$, or $90^\circ$ in the CW direction. Fig. F-3(a) shows that, if the $(-1, j0)$ point is to the left of the intersect of $L(j\omega)$ with the real axis, $\Phi_{11}$ is indeed $-90^\circ$. On the other hand, if the $(-1, j0)$ point is to the right of the intersect, as shown in Fig. F-3(b), when the value of $K$ is greater than 240, then $\Phi_{11}$ is $+270^\circ$, or, (more easily observed when the critical point is enclosed), the system would be unstable. Substituting $\Phi_{11} = 270^\circ$ into Eq. (F-16), we get $Z = 2$, which means that the characteristic equation has two roots in the right-half $s$-plane. Thus, for systems with minimum-phase loop transfer functions, the Nyquist

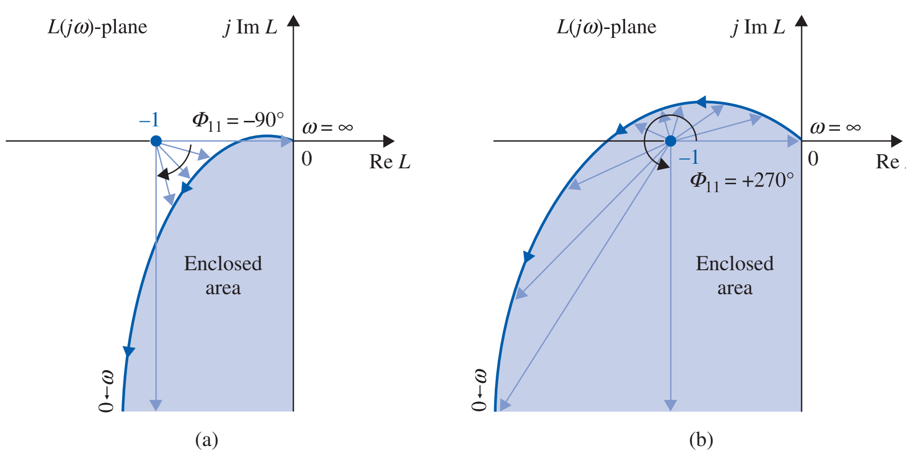


**Figure F-3** Nyquist plot of $L(s) = \dfrac{K}{s(s+2)(s+10)}$. (a) $K < 240$. (b) $K > 240$.

criterion with the "enclosure" test is easier to observe, but when the system is unstable, it does not tell how many characteristic equation roots are in the right-half plane; the general Nyquist criterion does.

▶ **EXAMPLE F-2-3** Consider that a control system has the loop transfer function

$$L(s) = \frac{K(s-1)}{s(s+1)} \tag{F-17}$$

We observed from the last equation that $P_\omega = 1$, and $P = 0$. The function $L(s)$ is of the nonminimum-phase type, because it has a zero at $s = 1$. Thus, the Nyquist criterion on enclosure cannot be used adequately in this case. From Eq. (F-8), the requirement for closed-loop stability is

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = -90^\circ \tag{F-18}$$

Thus, the stability criterion requires that the phasor drawn from the $(-1, j0)$ point to the Nyquist plot of $L(j\omega)$ should traverse $-90^\circ$ as $\omega$ varies from $\infty$ to 0.
To sketch the Nyquist plot of $L(s)$ that corresponds to the postive portion of the $j\omega$-axis of the $s$-plane, we set $s = j\omega$ in Eq. (F-17). We get

$$L(j\omega) = \frac{K(j\omega - 1)}{j\omega(j\omega + 1)} = \frac{K(j\omega - 1)}{-\omega^2 + j\omega} \tag{F-19}$$

When $\omega = \infty$,

$$L(j\infty) = \frac{K}{j\omega}_{\omega=\infty} = 0\angle -90^\circ \tag{F-20}$$

When $\omega = 0$,

$$L(j0) = \frac{K}{j\omega}_{\omega=0} = \infty\angle 90^\circ \tag{F-21}$$

To find the intersect of the $L(j\omega)$ plot on the real axis, we rationalize the function by multiplying the numerator and the denominator of Eq. (F-19) by $-\omega^2 - j\omega$. We have

$$L(j\omega) = \frac{K(j\omega - 1)(-\omega^2 - j\omega)}{\omega^4 + \omega^2} = \frac{K[2\omega + j(1-\omega^2)]}{\omega(\omega^2 + 1)} \tag{F-22}$$

Setting the imaginary part of $L(j\omega)$ to zero and solving for $\omega^2$, we have

$$\omega^2 = \pm 1 \text{ rad/sec} \tag{F-23}$$

For $\omega = 1$,

$$L(j1) = K \tag{F-24}$$

◀

**Nyquist Plot for $K > 0$**

Based on the preceding information, the Nyquist plot of $L(j\omega)$ that corresponds to the positive portion of the $j\omega$-axis is sketched as shown in Fig. F-4 for $K > 0$. Fig. F-4 shows that, as $\omega$ varies from $\infty$ to 0 along the Nyquist plot, the net angle $\Phi_{11}$ traversed by the phasor drawn from the $(-1, j0)$ point to the Nyquist plot is $+90^\circ$. Thus, the system is unstable because $\Phi_{11}$ is positive. We can also readily see that $(-1, j0)$ is enclosed by the Nyquist plot, so the same conclusion on closed-loop stability can be drawn.

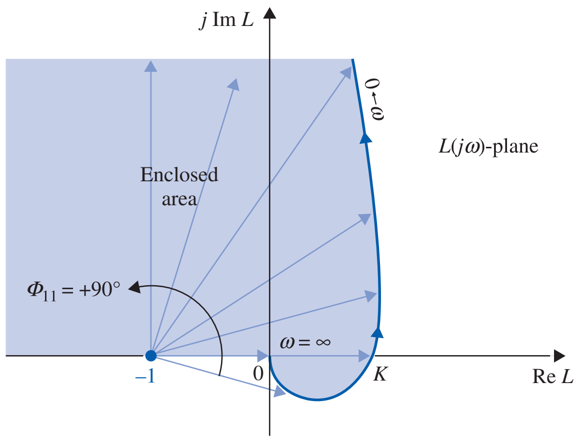


**Figure F-4** Nyquist plot of the system in Example F-2-3. $L(s) = \dfrac{K(s-1)}{s(s+1)}$.

From Eq. (F-7),

$$\Phi_{11} = (Z - 0.5P_\omega - P)180^\circ = (Z - 0.5)180^\circ = 90^\circ \tag{F-25}$$

Thus, $Z = 1$, which means that the characteristic equation of the closed-loop system has one root in the right-half $s$-plane. The characteristic equation of the system is

$$s^2 + (1 + K)s - K = 0 \tag{F-26}$$

We can easily verify that stability requires

$$0 > K > -1 \tag{F-27}$$

**Nyquist Plot for $K < 0$**

Fig. F-5(a) shows the Nyquist plot of $L(j\omega)$ when $K$ lies between 0 and $-1$. Notice that the plot is obtained by *rotating* the $L(j\omega)$ plot of Fig. F-4 by 180° about the origin. As $\omega$ is varied from $\infty$ to 0, the angle $\Phi_{11}$ in Fig. F-5(a) has a net rotation of $-90^\circ$, which agrees with the stability requirement in Eq. (F-18), and the closed-loop system is stable. It should be reminded that since $L(j\omega)$ is of the minimum-phase type, the fact that the Nyquist plot of $L(j\omega)$ of Fig. F-5(a) does not enclose the $(-1, j0)$ is *not* the reason that the system is stable.

Figure F-5(b) shows the Nyquist plot when $K < -1$. Now we see that the $(-1, j0)$ point is enclosed, so the system is unstable. Checking the value of $\Phi_{11}$, we have $\Phi_{11} = 270^\circ$, which differs from the required $-90^\circ$. Using Eq. (F-8),

$$\Phi_{11} = (Z - 0.5P_\omega - P)180^\circ = (Z - 0.5)180^\circ = 270^\circ \tag{F-28}$$

Thus, $Z = 2$, which means that the characteristic equation has two roots in the right-half $s$-plane.

In general, when $K$ changes signs, it is not necessary to redraw the Nyquist plot as shown in Fig. F-5. Eq. (8-40) can be written as

$$1 + L(s) = 1 + KL_1(s) = 0 \tag{F-29}$$

where $K$ is positive. For negative $K$, the last equation can be written as

$$1 - KL_1(s) = 0 \tag{F-30}$$

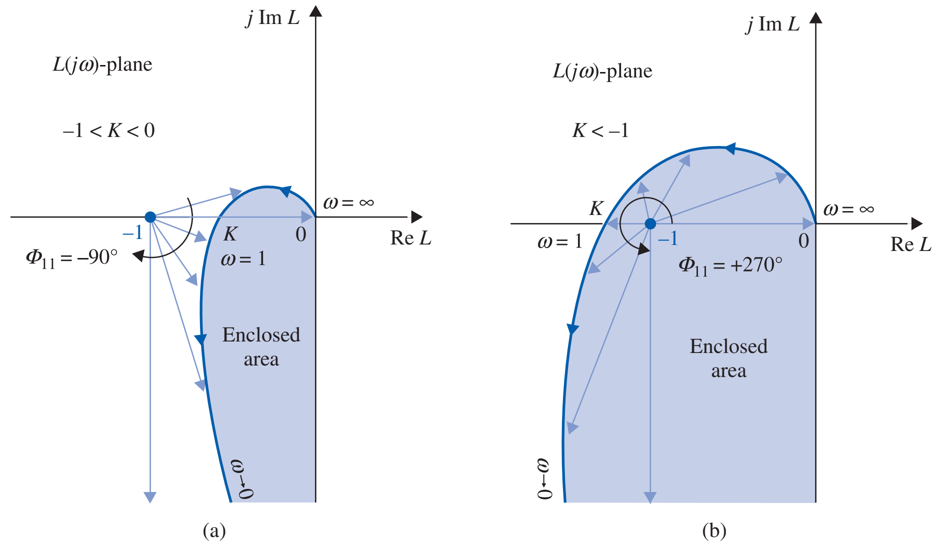


**Figure F-5** Nyquist plots of the system in Example F-2-3. $L(s) = \dfrac{K(s-1)}{s(s+1)}$. (a) $-1 < K < 0$. (b) $K < -1$.

or

$$KL_1(s) = 1 \tag{F-31}$$

where $K$ is now positive. Thus, Eq. (F-31) shows that, when $K$ is negative, we can still use the $L(j\omega)$ plot for positive $K$ but designate the $(+1, j0)$ point as the critical point for stability analysis.

Fig. F-6 shows the Nyquist plot of Eq. (F-17) for $K > 0$. When $K$ is negative, the $(+1, j0)$ point is regarded as the critical point. As shown in Fig. F-6, for $-1 < K < 0$, $\Phi_{11}$ is $-90°$, which is the required value, and the system is stable. When $K < -1$, the $(+1, j0)$ point is enclosed by the Nyquist plot, and the system is unstable. These results agree with those obtained from Fig. F-5 when the Nyquist plots for $K < 0$ were actually constructed.

It is of interest to compare the Nyquist stability analysis with the root-locus analysis. Fig. F-7 shows the root loci of the characteristic equation of the system with the loop transfer function given in Eq. (F-17). The stability condition of the system as a function of

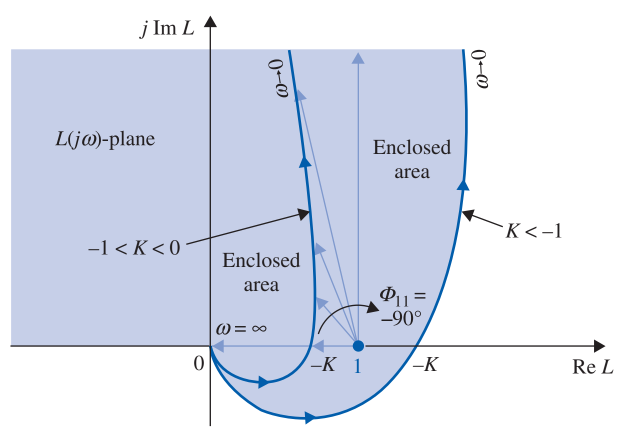


**Figure F-6** Nyquist plot of $L(s) = \dfrac{K(s-1)}{s(s+1)}$ with $K < 0$. The $(+1, j0)$ point is the critical point.

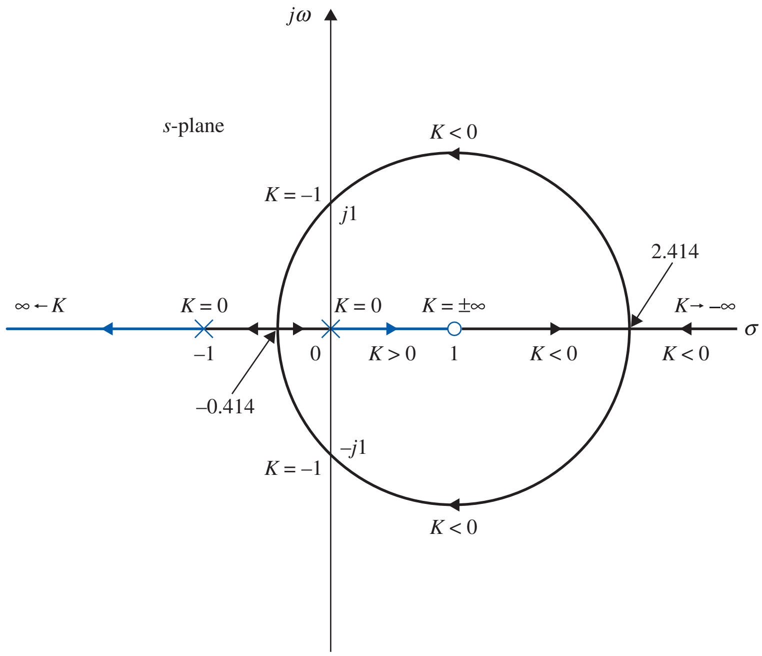


**Figure F-7** Complete root loci of system in Example F-2-3.

$K$ is clearly indicated by the root-locus diagram. The RL between 0 and +1 on the real axis indicates that the system is unstable for $0 < K < \infty$. The RL indicates that, for negative values of $K$, the system is unstable for $-\infty < K < -1$, and the system is stable only for the range of $-1 < K < 0$. The root loci cross the $j\omega$-axis at $\omega = \pm 1$ rad/sec, which are the values of $\omega$ at which the Nyquist plot of $L(j\omega)$ intersects the negative real axis.

**▶ EXAMPLE F-2-4** Consider the control system shown in Fig. F-8. It is desired to determine the range of $K$ for which the system is stable. The loop transfer function of the system is

$$L(s) = \frac{Y(s)}{E(s)} = \frac{10K(s+2)}{s^3 + 3s^2 + 10} \tag{F-32}$$

The poles of $L(s)$ are found to be at $s = -3.72$, $0.361 + j1.6$, and $0.361 - j1.6$. Or, we can use the Routh-Hurwitz criterion to verify that $L(s)$ has two poles in the right-half $s$-plane. Thus, $P = 2$, and $P_\omega = 0$. The transfer function $L(s)$ is of the nonminimum-phase type.

From Eq. (F-9), the requirement for the closed-loop system to be stable is

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = -360^\circ \tag{F-33}$$

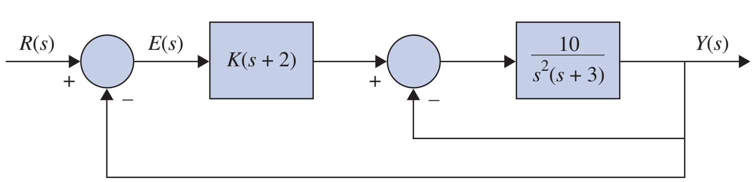


**Figure F-8** Block diagram of the control system in Example F-2-4.

Setting $s = j\omega$, Eq. (F-32) becomes

$$L(j\omega) = \frac{10K(j\omega + 2)}{(10 - 3\omega^2) - j\omega^3} \tag{F-34}$$

At $\omega = \infty$,

$$L(j\infty) = 0\angle 180^\circ \tag{F-35}$$

At $\omega = 0$,

$$L(j0) = 2K \tag{F-36}$$

To find the intersect on the real axis of the $L(j\omega)$-plane, we rationalize $L(j\omega)$ as

$$L(j\omega) = \frac{10K\left\{2(10 - 3\omega^2) - \omega^4 + j[\omega(10 - 3\omega^2) + 2\omega^2]\right\}}{(10 - 3\omega^2)^2 + \omega^6} \tag{F-37}$$

Setting the imaginary part of $L(j\omega)$ to zero, we have

$$\omega\left(10 - 3\omega^2\right) + 2\omega^3 = 0 \tag{F-38}$$

The solutions of the last equation are $\omega = 0$ and $\omega = \pm\sqrt{10} = 3.16$ rad/sec, which are the frequencies at which the $L(j\omega)$ plot intersects the real axis of the $L(j\omega)$-plane. When $\omega = 0$, we already have $L(j0) = 2K$ in Eq. (F-36). When $\omega = 3.16$ rad/sec,

$$L(j3.16) = -K \tag{F-39}$$

Fig. F-9(a) shows the Nqyuist plot of $L(j\omega)$ for $0 < K < 1$. Because the $(-1, j0)$ point is enclosed by the Nyquist plot, the closed-loop system is unstable. We can also show that the angle traversed by $\Phi_{11}$ is $0^\circ$, not $-360^\circ$, as required in Eq. (F-33). Fig. F-9(b) shows the Nyquist plot of $L(j\omega)$ when $K$ is greater than unity. In this case, the angle $\Phi_{11}$ rotates a total of $-360^\circ$; thus, the system is stable.

When $K$ is negative, we can use the plots in Fig. F-9 and regard the $(+1, j0)$ point as the critical point. The following stability conditions are observed:

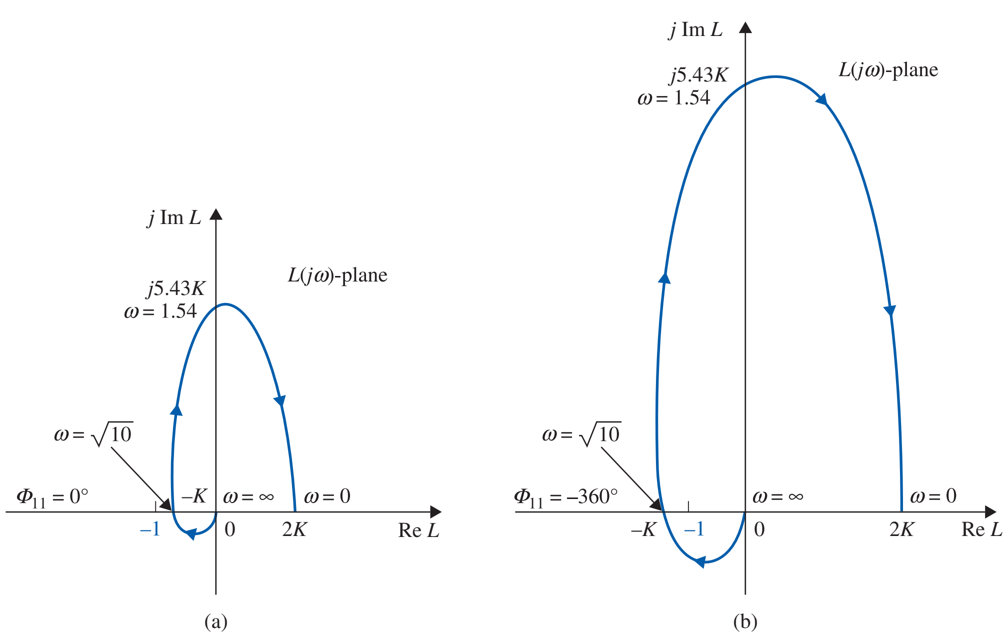


**Figure F-9** Nyquist plots of the system in Example F-2-4. (a) $1 > K > 0$. (b) $K > 1$.

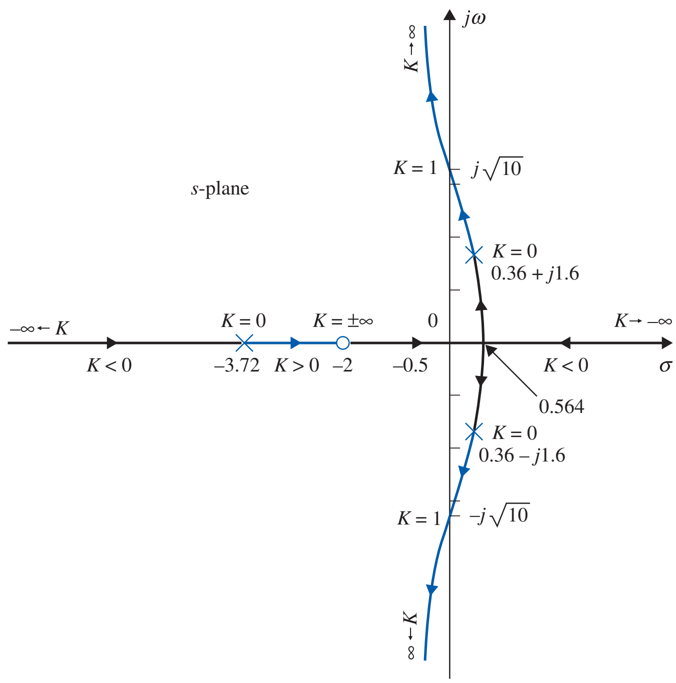


**Figure F-10** Complete root loci of the system in Example F-2-4.

- $2K < -1$: The $(+1, j0)$ point lies between 0 and $2K$ and is not enclosed by $L(j\omega)$, but $\Phi_{11} = -180°$; the system is unstable. For stability, $\Phi_{11}$ must equal $-360°$.
- $-1 < 2K < 0$: The $(+1, j0)$ point is to the right of the point $2K$ and is enclosed by the Nyquist plot; the system is unstable. In this case, $\Phi_{11} = 0°$.

The conclusion is that the system is stable for $K > 1$. The root loci of the system are shown in Fig. F-10. Clearly, when $K$ is negative, one branch of the RL will always stay in the right-half plane, and the system is unstable. The system is stable only for $K > 1$, and the root loci cross the $j\omega$-axis at $\omega = \pm 3.16$ rad/sec, which corresponds to the frequency at which the $L(j\omega)$ plot intersects the negative real axis. The value of $K$ at the crossing point on the $j\omega$-axis is 1. ◀

▶ **EXAMPLE F-2-5** Consider a control system with the loop transfer function

$$L(s) = \frac{K}{(s+2)(s^2+4)} \tag{F-40}$$

which has a pair of imaginary poles at $s = j2$ and $-j2$. Thus, $P_\omega = 2$, and $P = 0$. To apply the Nyquist criterion in the original form, we would have to define the Nyquist path with small indentations around these poles.

Instead of constructing the entire Nyquist plot, the portion that corresponds to $s = j\infty$ to $j0$ is plotted as shown in Fig. F-11. The data for this Nyquist plot are easily obtained using any of the frequency-domain programs mentioned earlier.

From Eq. (F-9), the value of $\Phi_{11}$ required for stability is

$$\Phi_{11} = -(0.5P_\omega + P)180° = -180° \tag{F-41}$$

As seen from Fig. F-11, the magnitude of $L(j\omega)$ goes to infinity when $\omega = 2$ rad/sec. When $K$ is positive, the critical point $(-1, j0)$ is not enclosed by the Nyquist plot, and the system is unstable. For the angle check,

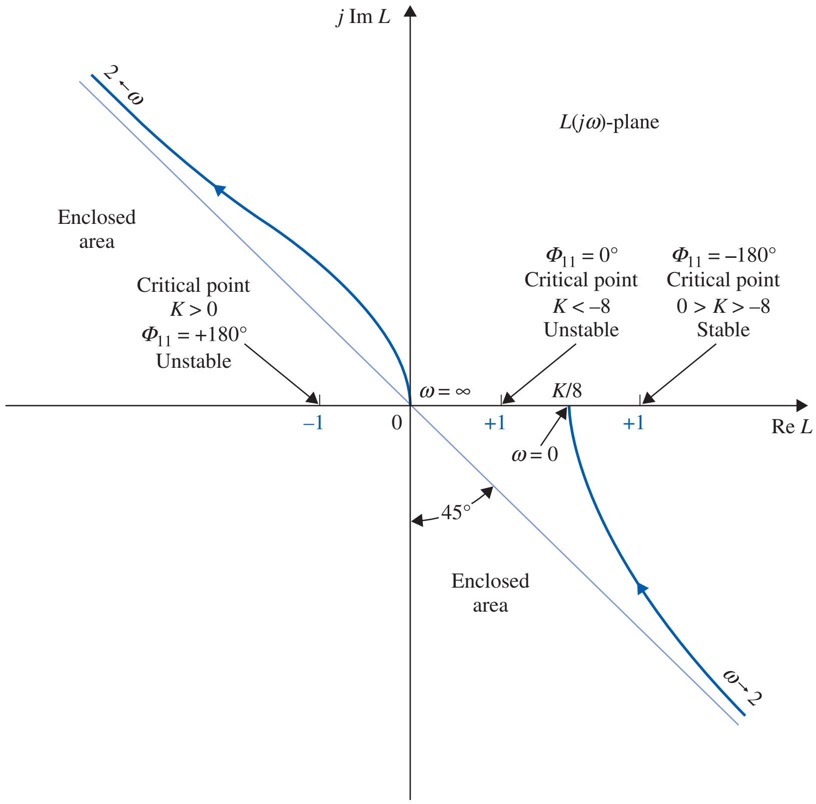


**Figure F-11** Nyquist plot of the control system in Example F-2-5.

when $\omega$ varies from $\infty$ to 2, the angle $\Phi_{11}$ is $+135°$, and for the portion of $\omega = 2$ to 0, $\Phi_{11}$ is $+45°$. Thus, the total $\Phi_{11}$ is $+180°$, not $-180°$. The system is unstable for all positive values of $K$.

When $K$ is negative, the critical point in Fig. F-11 is at $(+1, j0)$. Fig. F-11 shows that, if the $(+1, j0)$ point lies between 0 and $K/8$, it is enclosed by the Nyquist path, and the system is unstable. Thus, the system is unstable for $K < -8$. When the $(+1, j0)$ point is to the right of the $K/8$ point, $\Phi_{11}$ from $\omega = \infty$ to $\omega = 2$ is $-45°$, and from $\omega = 2$ to $\omega = 0$ is $-135°$. Thus the total $\Phi_{11}$ as $\omega$ varies from $\infty$ to 0 is $-180°$, which agrees with the value required in Eq. (F-41). The system is stable for $0 > K > -8$. The summary of the Nyquist criterion application to this system is as follows.

| Range of $K$ | $\Phi_{11}$ (deg) for $\omega = 2$ to 0 | $\Phi_{11}$ (deg) for $\omega = 2$ to $\infty$ | Total $\Phi_{11}$ (deg) | Critical Point | Stability Condition |
|---|---|---|---|---|---|
| $K > 0$ | +135 | +45 | +180 | −1 point enclosed | Unstable |
| $K < -8$ | −45 | +45 | 0 | +1 point enclosed | Unstable |
| $-8 < K < 0$ | −45 | −135 | −180 | | Stable |

The complete root loci of the characteristic equation of the system are constructed in Fig. F-12 using the pole–zero configuration of Eq. (F-40). The stability condition of $-8 < K < 0$ is easily viewed from the root loci.

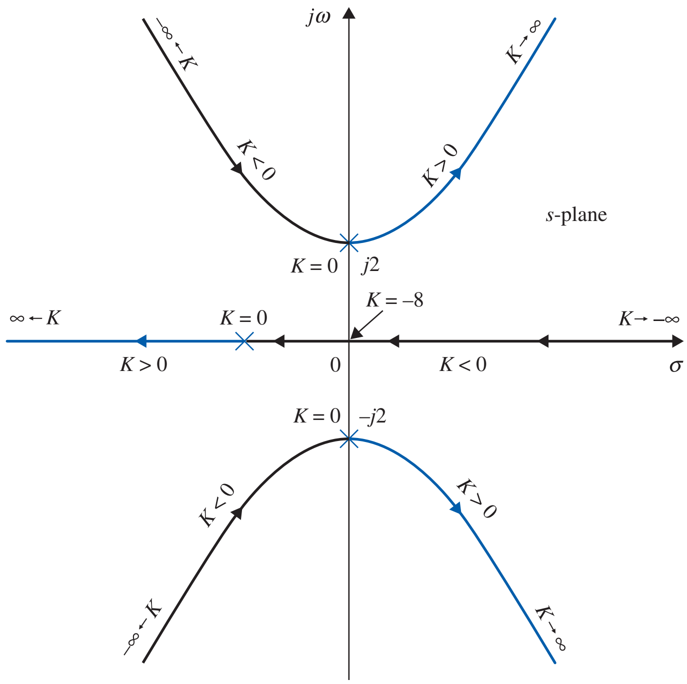


**Figure F-12** Complete root loci of the system in Example F-2-5.

## ▶ F-3 STABILITY ANALYSIS OF MULTILOOP SYSTEMS

The Nyquist stability analyses conducted in the preceding sections are all directed toward the loop transfer function $L(s)$. It does not matter whether the system is with single loop or multiple loops, because once the loop transfer function is obtained, stability analysis can be conducted using the Routh-Hurwitz criterion, root loci, or the Nyquist criterion.

For multiloop-feedback systems, it may be advantageous to analyze the stability of the system by working from the inner loop toward the outer loop, one at a time. This way, more insight may be gained on the stability of the individual loops of the system. The following example will illustrate this approach.

**▶ EXAMPLE F-3-1** Fig. F-13 shows the block diagram of a system that controls the gun turret of a tank. During servicing of the turret control system, the mechanic accidentally opened the outer loop of the system. With the power turned on, the gun turret went out of control and finally flew apart. The purpose of this example is to show that it is inadequate to investigate just the stability of the overall system. In general, for a multiloop control system, one should conduct a systematic stability analysis of all the inner loops of the system. It is admissible to have unstable inner loops, as long as the overall system is stable. However, if such a situation exists, it is important to forewarn or take precautionary measures to prevent opening the loops during operation.

The loop transfer function of the inner loop is

$$G(s) = \frac{6}{s(s+1)(s+2)} \tag{F-42}$$

Fig. F-14 shows the Nyquist plot of $G(s)$. Because the plot intersects the real axis at the $-1$ point at $\omega = 1.414$ rad/sec, the inner loop is marginally stable. Therefore, if the outer loop of the system is opened, the system will oscillate continuously with a frequency of 1.414 rad/sec. The loop transfer

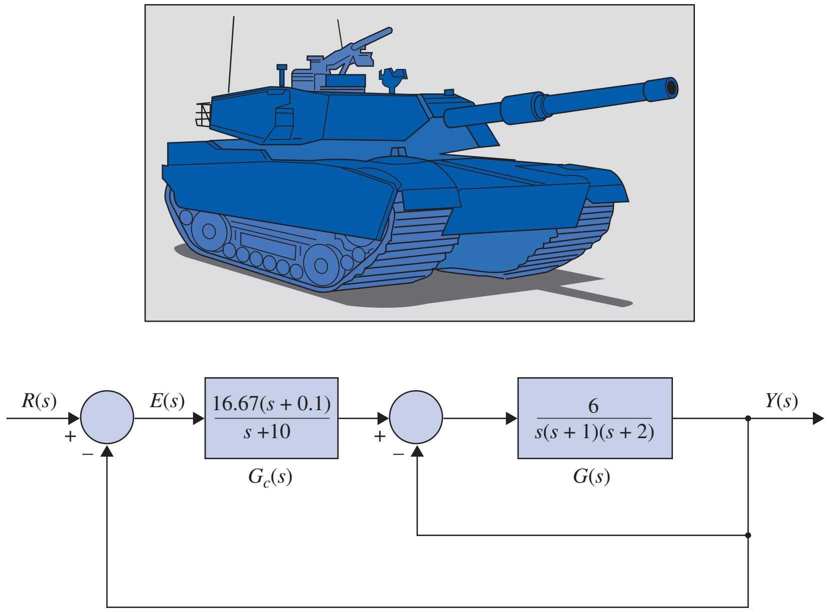


**Figure F-13** Multiloop feedback control system for tank turret control.

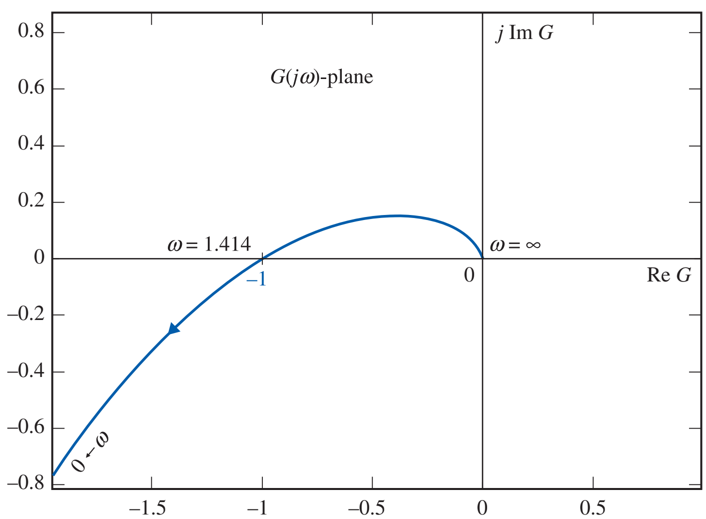


**Figure F-14** Nyquist plot of $G(s) = \dfrac{6}{s(s+1)(s+2)}$.

function of the overall system is

$$L(s) = \frac{G_c(s)G(s)}{1+G(s)} = \frac{100(s+0.1)}{(s+10)(s^3+3s^2+2s+6)} \tag{F-43}$$

Because $L(s)$ has two poles on the $j\omega$-axis and the rest are in the left-half $s$-plane, $P_\omega = 2$ and $P = 0$. The function is also of the nonminimum-phase type, so we must use Eq. (F-9) for the stability test of the overall system. Thus,

$$\Phi_{11} = -(0.5P_\omega + P)180^\circ = -180^\circ \tag{F-44}$$

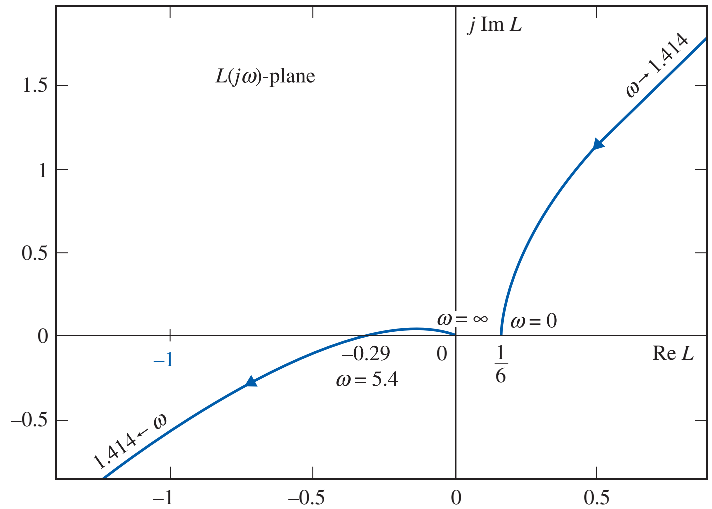


**Figure F-15** Nyquist plot of $L(s) = \dfrac{100(s+0.1)}{(s+10)(s^3+3s^2+2s+6)}$.

The Nyquist plot of $L(s)$ is plotted as shown in Fig. F-15. The angle $\Phi_{11}$ for $\omega = \infty$ to $\omega = 1.414$ rad/sec is $-90°$, and from $\omega = 1.414$ rad/sec to $\omega = 0$ is $-90°$. Thus, the total value of $\Phi_{11}$ for $\omega = \infty$ to $\omega = 0$ is $-180°$, and the overall system is stable.

In general, when more than two loops are involved, the proper way is to start with the stability of the innermost loop by opening all the outer loops, and then add one loop at a time, until the outermost loop is closed. ◀

## ▶ PROBLEMS

**F-1.** The loop transfer functions $L(s)$ of single-feedback-loop systems are given in the following equations. Sketch the Nyquist plot of $L(j\omega)$ for $\omega = 0$ to $\infty$. Determine the stability of the closed-loop system. If the system is unstable, find the number of poles of the closed-loop transfer function that are in the right-half $s$-plane. Solve for the intersect of $L(j\omega)$ on the negative real axis of the $L(j\omega)$-plane analytically. You may construct the Nyquist plot of $L(j\omega)$ using any computer program.

**(a)** $L(s) = \dfrac{5(s-2)}{s(s+1)(s-1)}$ **(b)** $L(s) = \dfrac{50}{s(s+5)(s-1)}$

**(c)** $L(s) = \dfrac{3(s+2)}{s(s^3+3s+1)}$ **(d)** $L(s) = \dfrac{100}{s(s+1)(s^2+2)}$

**(e)** $L(s) = \dfrac{s^2-5s+2}{s(s^3+2s^2+2s+10)}$ **(f)** $L(s) = \dfrac{-0.1(s^2-1)(s+2)}{s(s^2+s+1)}$

**F-2.** The loop transfer functions of single-feedback-loop control systems are given in the following equations. Apply the Nyquist criterion and determine the values of $K$ for the system to be stable. Sketch the Nyquist plot of $L(j\omega)$ with $K = 1$ for $\omega = 0$ to $\omega = \infty$. You may use a computer program to plot the Nyquist plots.

**(a)** $L(s) = \dfrac{K(s-2)}{s(s^2-1)}$ **(b)** $L(s) = \dfrac{K}{s(s+10)(s-2)}$

**(c)** $L(s) = \dfrac{K(s+1)}{s(s^3+3s+1)}$ **(d)** $L(s) = \dfrac{K(s^2-5s+2)}{s(s^3+2s^2+2s+10)}$

**(e)** $L(s) = \dfrac{K(s^2 - 1)(s + 2)}{s(s^2 + s + 1)}$ **(f)** $L(s) = \dfrac{K(s^2 - 5s + 1)}{s(s + 1)(s^2 + 4)}$

**F-3.** Fig. FP-3 shows the Nyquist plots of the loop transfer function $L(j\omega)$ for $\omega = 0$ to $\omega = \infty$ for single-feedback-loop control systems. The number of poles of $L(j\omega)$ that are on the $j\omega$-axis, $P_\omega$, and in the right-half $s$-plane, $P$, are indicated for each case. Determine the stability of the closed-loop system by applying the Nyquist criterion. For the unstable systems, give the number of zeros of $1 + L(s)$ that are in the right-half $s$-plane.

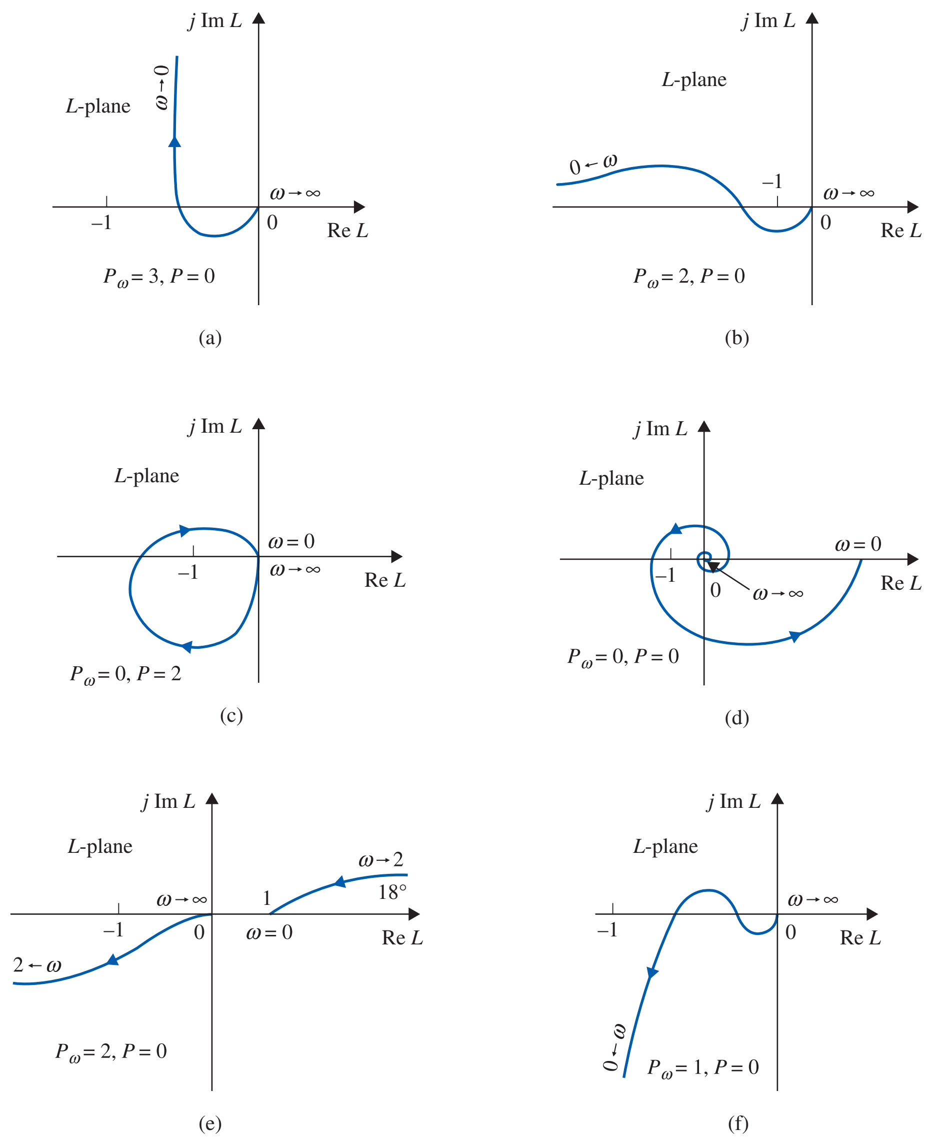


**Figure FP-3**

**F-4.** It was mentioned in the text that, when the function $L(j\omega)$ has more zeros than poles, it is necessary to plot the Nyquist plot of $1/L(j\omega)$ to apply the simplified Nyquist criterion. Determine the stability of the systems described by the function $1/L(j\omega)$ shown in Fig. FP-4. For each case, the values of $P_\omega$ and $P$ for the function $1/L(j\omega)$ are given, where $P_\omega$ refers to the number of poles of $1/L(j\omega)$ that are on the $j\omega$-axis, and $P$ refers to the number of poles of $1 + 1/L(j\omega)$ that are in the right-half $s$-plane.

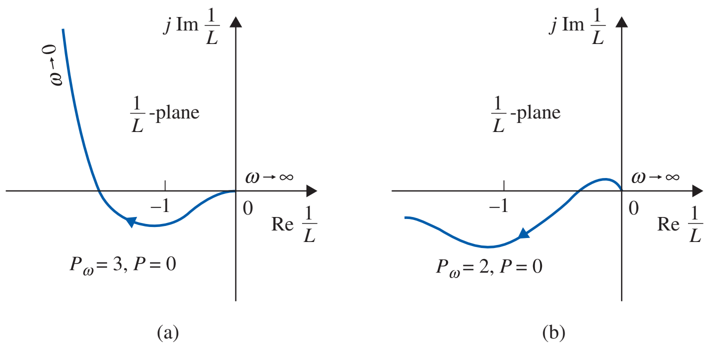


**Figure FP-4**

**F-5.** Fig. FP-5 shows the Nyquist plots of the loop transfer function $L(j\omega)$ for $\omega = 0$ to $\omega = \infty$ for single-feedback-loop control systems. The gain $K$ appears as a multiplying factor in $L(s)$. The number of poles of $L(j\omega)$ that are on the $j\omega$-axis and in the right-half $s$-plane are indicated in each case. Determine the range(s) of $K$ for closed-loop system stability.

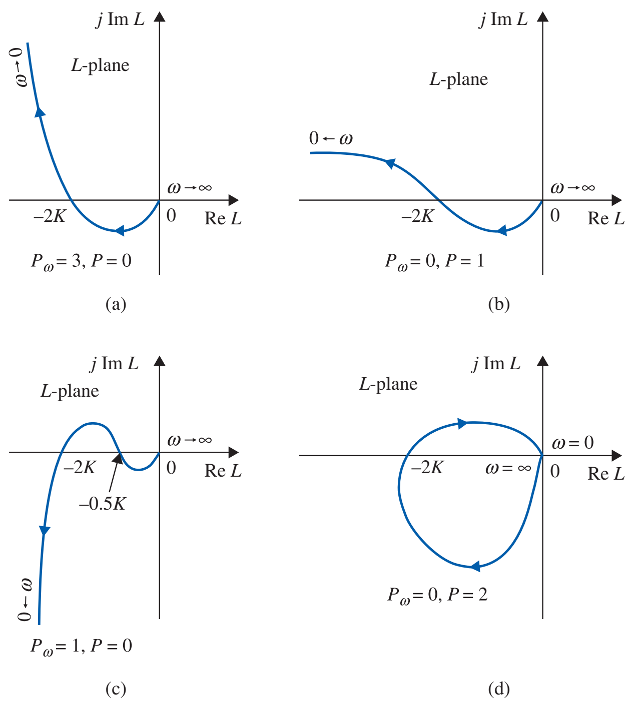


**Figure FP-5**

**F-6.** The characteristic equations of linear control systems are given below. Apply the Nyquist criterion to determine the values of $K$ for system stability. Check the answers by means of the Routh-Hurwitz criterion.

**(a)** $s^3 + 4Ks^2 + (K + 5)s + 10 = 0$

**(b)** $s^3 + K(s^3 + 2s^2 + 1) = 0$

**(c)** $s(s + 1)(s^2 + 4) + K(s^2 + 1) = 0$

**(d)** $s^3 + 2s^2 + 20s + 10K = 0$

**(e)** $s(s^3 + 3s + 3) + K(s + 2) = 0$

## ▶ REFERENCES


1. K. S. Yeung, "A Reformulation of Nyquist's Criterion," *IEEE Trans. Educ.*, Vol. E-28, pp. 58–60, Feb. 1985.